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ISSUE No. 7

PRACTICAL MATHEMATICS

THEORY AND PRACTICE WITH MILITARY AND INDUSTRIAL APPLICATIONS

TRIGONOMETRY

Elements of Trigonometry

Angles and Measurements
Trigonometric Functions
Right and Oblique Triangles
Identities
Multiple Angles
Inverse Functions

The Slide Rule in Trigonometry

—ALSO—

Mathematical Tables and Formulas
Glossary of Mathematical Terms
Self-Tests and Trigonometry Problems

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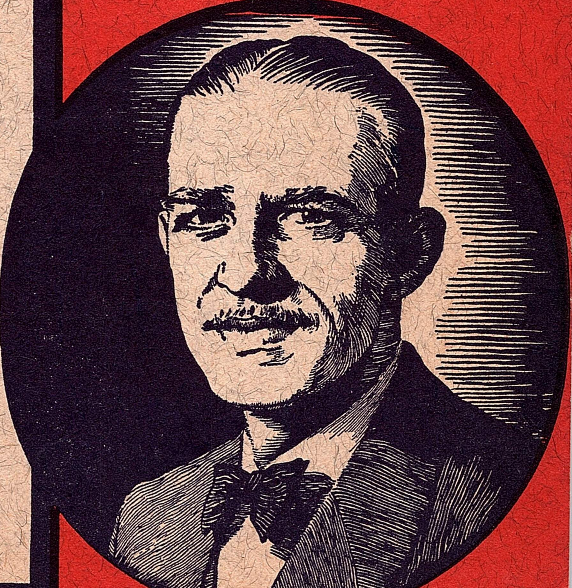

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EDITOR: REGINALD STEVENS KIMBALL ED.D.

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CHATS WITH THE EDITOR

WITH this issue of PRACTICAL MATHEMATICS, we reach the half-way mark in our course in basic mathematics for military and industrial workers. Thus far, we have reviewed arithmetic and algebra, and have made a rapid survey of plane and solid geometry. In the six issues which have preceded, we have laid the ground-work for understanding selected topics from higher mathematics.

When we turn to trigonometry, we reach a field which used to be reserved for the specialist. Up to a few years ago, many students did not encounter trigonometry until they reached the college campus. The many whose formal education terminated with graduation from high school usually had only vague conceptions of what trigonometry was all about, as such courses in "trig" as were offered at the high-school level were reserved for the more proficient students.

Trigonometry offers one of the most interesting fields for an extension of mathematical knowledge, since the solution of problems involving triangles is intimately related to many phases of navigation, aviation, gunnery, and machine-shop practice. This issue is, then, an essential stepping-stone toward the work in applied mathematics to which we shall devote the last four issues of the present course.

Within the past decade or so, fortunately for the war effort, it has become customary to introduce junior-high-school students to a few basic principles in trigonometry. Most of the modern textbooks in algebra and plane geometry contain a section in which a bit of trigonometry is introduced. Courses in shop mathematics, offered in the vocational schools, usually contain a few elements of trigonometry, and charts showing the relationship of sides and angles of triangles frequently adorn the walls of shop classrooms.

In the present issue, we are presenting enough trigonometry to enable the average person to utilize the functional values in the solution of practical problems. Dr. Agnew has selected the essential facts and formulas of trigonometry and has presented them in a simple, non-technical fashion. At various points in his article, he warns the reader not to proceed past that point until he has made certain that he has a working knowledge of what has gone before. I hope that you will pay due heed to these warnings, as your continued advancement in the use of the more complex formulas depends upon your having mastered the treatment of the earlier sections.

Once you have discovered just what trigonometry is all about, you will probably begin to ask yourself why it has been kept away from you heretofore. As you attempt to solve the practical problems which follow each section, you will see more clearly that, without some trigonometry, we should be unable to solve many problems which become "perfectly simple" when we call in its aid.

As we have pointed out to you many times before, logarithms will save you much time and effort in the numerical computations which are a necessary part of any mathematical

exercise. If you form the habit of using the "log" tables given in Issue Number Two and the new logarithmic tables of trigonometric functions which will be found on pages 442 to 447 of this issue, you will relieve yourself of the burden of computation and thus be able to concentrate your attention on the selection of the proper formula to be used in solving your problem.

In the present issue, we are devoting more pages to tables than has been necessary in previous issues. Even the abridged tables of trigonometric functions and their logarithms of numbers (Issue Number Two, pages 126 and 127), giving the values of the functions at intervals of fifteen minutes for angles ranging from one to ninety degrees, are sufficiently definite to assist you in solving to four significant figures any problem which you are likely to encounter. If you desire greater accuracy, you should turn to the more complete tables to which we have made reference before (Issue Number Two, page 97).

The slide rule, also, may be brought into play in the solution of many problems involving trigonometry. Subject to the same limitations of accuracy as were discussed in the previous article on the slide rule (Issue Number Two, pages 101 to 114), the "trig" slides on the ordinary ten-inch slide rule give you a means of quick computation. In the present issue, we devote an article to the use of the "trig" panel on the reverse of the slide.

In the tables this time, you will find (page 441) a list of the letters in the Greek alphabet. Because trigonometry makes so much use of angles, we find it easier on many occasions to refer to an angle by giving it the name of a single letter rather than to name it by using three letters, as in plane geometry. Since we use capital Roman letters to refer to points (and combinations of these to refer to lines and figures) and since we use lower-case Roman letters to refer to lengths of lines (and combinations of these to refer to areas and volumes), we should be confused if we brought either into play in referring to the angles. For this reason, mathematicians have adopted the device of referring to angles by use of lower-case Greek letters. You do not have to master the entire Greek alphabet. You probably struck up an acquaintance with π (designating the relationship between radius and circumference of the circle) back in the elementary school. For referring to "any angle", we customarily use θ or φ, just as we use x and y for the unknown in algebra. In the designation of the angles of a triangle, we match the Greek letter with the Roman letter which we have used to denote the side of the triangle which is opposite the angle, α being the angle opposite side a, β the angle opposite side b, and (since the Greek alphabet has no letter corresponding to the Roman C) γ the angle opposite side c. Certain of the other Greek letters (both capital and lower case) will be introduced in subsequent issues in connection with formulas. The table will help you to learn the proper designation for each as you encounter it.

Because of the additional space devoted to tables, we are omitting from this issue some of the customary departments, such as "Odd Problems for Off Hours". These will be resumed in the next issue.

I wish that it were possible to answer individually all of the letters which some of you have been sending in. We had realized that there was need for a magazine of the sort that we intend PRACTICAL MATHEMATICS to be, but we had not anticipated that the demand would be so great. Naturally, we are gratified that you are finding PRACTICAL MATHEMATICS so

helpful to you in the solution of your individual problems.

With high schools and colleges concentrating attention on mathematics to a degree never before reached in the history of American education, it is not surprising that adult workers in the war industries and young men and women who are aspiring officer-candidates in the military and naval forces should also wish to acquire greater facility in the use of mathematics. By selecting just the sort of problem which these people will meet in the course of their daily work, we are offering, in the pages of PRACTICAL MATHEMATICS, an opportunity to put to immediate use the mathematical principles which we are teaching.

It is now something like sixty days since the first issue of PRACTICAL MATHEMATICS came to you. If you have been working faithfully on the topics and have been finishing each issue within the ten-day period before the next issue comes along, you will have mastered by this time a substantial part of mathematical theory. Some of our readers tell us that they have been keeping up in this fashion. Others, having only an hour or so a day to devote to the subject, are finding that they cannot complete one issue before the next arrives.

That is, of course, an individual problem. There is no compulsion upon any one of you to get an issue completely out of the way before the next arrives. You are your own "boss" as to the rate of speed with which you progress. If you find that it takes you longer than ten days to complete an issue, simply set the new issues aside until you are ready for them. You will find that they are just as valuable a month from now (or even six months from now) as they were when they first reached you. It is more essential that you school yourself to complete understanding of one topic than that you forge ahead at a speed which means you are not understanding all that you are reading.

We are issuing the materials rapidly because some of our readers are desirous of having the complete course in their hands at as early a date as possible. Those who are already well grounded in the elementary branches of mathematics need to give only a quick review to those branches before proceeding with the new subject-matter. The Measuring Rod section of each issue gives an opportunity to judge the extent to which you understand the subject-matter of each issue. If you can solve the problems in a given issue, you obviously do not need to spend too much time in reviewing the articles there presented.

Are you following an orderly procedure in the solving of your problems? Many students create additional and unnecessary difficulties for themselves because they follow a haphazard method (or lack of method) in jotting down their computations and then find it difficult to check back over their figuring to locate possible errors. If you will lay out all of your work in systematic form, following the style of the illustrative examples printed in the text or in the workbook, you will have a work-sheet which will enable you to check your work more quickly. Where additional computations are necessary, it is a good procedure to label each "scribble" so that you may readily identify it as a part of your progression toward the final solution.

Especially in computations where you have occasion to add a long series of numbers, you will find that irregular, straggling columns, make it virtually impossible to add the proper numbers together. If you keep your figures in even columns, you eliminate this difficulty.

Before proceeding with a logarithmic solution, you may help your-

self to see the steps which are ahead if you first write the formula which you are going to employ, then write the same thing in log and colog form, and finally set in the numbers whose logarithms you wish to find. After these preliminary steps, you may look up all of the logarithms at one and the same time, noting their relationship to the numbers which they represent. In this way, you will be sure that nothing has escaped your attention.

In such a case, it is evident that you will want a side column in which you note the computation of cologarithms, so that you may check back over their solution. Having found the cologarithms first, you are the better able to select the figures you desire for the actual solution of the problem.

Once the problems are solved, you may wish to refer to them again when you find it necessary to solve a similar problem. If you keep your pages arranged in order, with each exercise identified by the numbers used in the text, you may build for yourself a notebook of key problems which proceeds in the same orderly fashion as the issues of the magazine. The index to the issues of Practical Mathematics will thus be an index also to your own set of solutions. If you are using our *Practical Mathematics Workbook*, secure for yourself some paper of the same size and insert the pages of your own work at the proper places along with the workbook pages on the same subject. We venture to predict that, in years to come, you will find this "fifteenth issue" (of your own making) one of the most valuable parts of the entire course.

In the next issue, the eighth, we shall select certain topics from the calculus which will be equally important for a mastery of the applied subjects. The calculus enables us to determine the constant relationships among certain values. Issue Number Eight will make clear to you the practical applications which you may make of some of the elementary principles of the calculus.

R. S. K.

ABOUT OUR AUTHORS

RALPH PALMER AGNEW was born at Boardman, Ohio, on December 29, 1900, just at the turn of the century. He received the degree of Bachelor of Arts from Allegheny College in 1923, the degree of Master of Science from Iowa State College in 1925, and the degree of Doctor of Philosophy from Cornell University in 1930.

From 1923 to 1925, he served as a graduate assistant at Iowa State and then went to Cornell as an instructor, spending the years from 1925 to 1930 in that capacity. As a National Research Fellow, he divided the years from 1930 to 1932 among the University of Cincinnati, Princeton University, and Brown University. In 1932, he returned to Cornell as Assistant Professor and was promoted to the rank of full Professor in 1938.

Last year, Dr. Agnew brought out a textbook on *Differential Equations*. He has published over 30 papers in various mathematical research journals. He is a member of the American Mathematical Society, the Mathematical Association of America, the American Association for the Advancement of Science, Pi Mu Epsilon, Phi Kappa Phi, and Sigma Xi.

· ELEMENTS OF TRIGONOMETRY ·

By Ralph P. Agnew, Ph.D.

A *trigon* (three-gon or three-angle or triangle) is a figure such as that shown in Fig. 1. The three points, A, B, and C, are called *vertices* of the triangle. The three line-segments, AB, BC, and CA, are the *sides* of the triangle. So much we learned in Issue Number Five, on plane geometry. The lengths of the sides may be measured in meters, inches, feet, miles, rods, furlongs, chains, or any other one of the many units of length with which the human race is blessed or cursed.

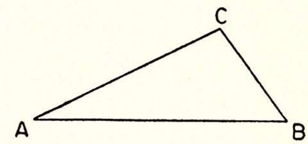

Fig. 1

The *angle* at A represents the amount of rotation necessary to turn AB about the point A in a counterclockwise direction until it becomes coincident with AC. The units of angle (or of rotation) with which we must presently become familiar are the following: *revolution*, *right angle*, *radian*, *degree*, *minute*, *second* and *mil.*

ANGLES AND MEASUREMENTS

The letters, *metry*, of the word, *trigonometry*, come from the Latin and Greek words, *metrum* and μετρον, meaning *measure.* Hence, trigonometry could be expected to be the science of measuring triangles. Actually, trigonometry is applied not only to triangles but also to many diverse problems in which angles or directions play a rôle. For example, if one is told that a gun is pointed in a direction 20 degrees north of east and that the barrel of the gun makes an angle of 30 degrees with the horizontal base of the gun, one would use trigonometry in the process of determining the path (*trajectory*) of a given shell fired from the gun. Again, if an electric generator is driven at a certain speed measured, say, in revolutions per second, one would use trigonometry to find the electric current generated. We shall be able to present some of the applications of trigonometry as we develop the subject. Many important applications can be appreciated only after one has completed his study of trigonometry and has taken up the

study of the calculus and topics in applied mathematics, in later issues of PRACTICAL MATHEMATICS.

Trigonometry is usually considered the most interesting of the mathematics courses which precede the calculus.

Generation of angles

Let the point, *O*, in Fig. 2 be regarded as the hub of a wheel, and let the line-segment, *OX*, be the initial position of a spoke which rotates as the wheel turns about the point, *O*. As the spoke rotates in a counterclockwise direction from the *initial position*, *OX*, to the *terminal position*, *OP*, it is said to *generate* the angle shown in Fig. 2. If the spoke rotates in a clockwise direction from the initial position, *OX*, to the terminal position, *OP*, it generates β, shown in Fig. 3.

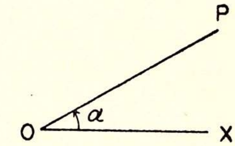

Fig. 2

In plane geometry, one sometimes uses the symbol, *O*, or the symbol, *XOP*, to denote the angle shown in Fig. 2. However, these notations do not distinguish between the angle generated in Fig. 2 and the angle generated in Fig. 3. A more intelligible notation is required. In trigonometry and its applications, the lower case (small, not capital) Greek letters are used to denote angles. (See Table XXXVIII, page 441.) The letters most frequently used are α (alpha), β (beta), γ (gamma), φ (phi), and θ (theta).

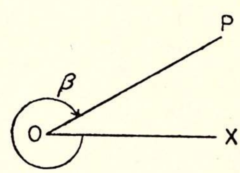

Fig. 3

For example, in Fig. 2, α is used to denote the angle through which a spoke of a wheel rotates in passing in the direction of the arrow (counterclockwise) from the initial position, *OX*, to the terminal position, *OP*. Similarly, in Fig. 3, β denotes the angle generated as the spoke passes in the direction of the arrow (clockwise) from the initial position, *OX*, to the terminal position, *OP*.

REVOLUTIONS AND RIGHT ANGLES

If a line-segment with one end at *O* (the spoke of a wheel) rotates in a counterclockwise direction from the initial position, *OX*, until it has turned completely around once and the terminal position accordingly coincides with the initial position, the angle γ of Fig. 4 is generated. It is natural to say that the angle generated is one revolution. Thus, the *revolution* is introduced naturally as a unit for measuring angles. To say that a blade of an airplane propeller turns through an angle of 3,000 revolutions each minute is meaningful and sensible.

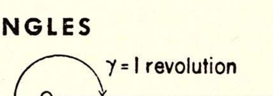

Fig. 4

A *right angle* (Fig. 5) is the angle between two perpendicular line-segments, as we learned in

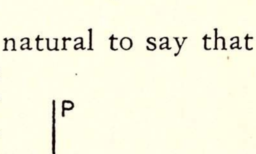

Fig. 5

plane geometry, page 259. It is easily seen that one revolution is equal to 4 right angles. This gives the first entry in Table XXXI (page 440), which furnishes the fundamental relations between the standard units for measurement of angles.

DEGREES

A *degree* is a unit of angle defined by one or the other of the two equivalent formulas,

$$360 \text{ degrees} = 1 \text{ revolution}$$

or

$$90 \text{ degrees} = 1 \text{ right angle.}$$

A *minute* is defined by the formula,

$$60 \text{ minutes} = 1 \text{ degree.}$$

The number of minutes in 360 degrees is then 360×60, or 21,600.

A *second* is defined by the formula,

$$60 \text{ seconds} = 1 \text{ minute.}$$

The number of seconds in one revolution is then $21{,}600 \times 60$, or 1,296,000.

It is, as the reader doubtless knows, customary to write 45 degrees in the form, $45°$; 20 minutes in the form, $20'$; and 35 seconds in the form, $35''$.

Illustrative Example

Find the number of degrees, minutes, and seconds through which the hour hand of a clock turns in 25 minutes.

The minute hand turns through $\frac{25}{60}$ of a revolution or 150°. The angle through which the hour hand turns is $\frac{1}{12}$ as great (Why?) and hence is 12.5°, or $12°30'0''$.

RADIANS

For reasons which the reader cannot fully appreciate at this time, the *radian* is the most natural and useful unit for measurement of angles. The radian is the unit used in the calculus and in all phases of pure and applied mathematics in which significant use of the calculus is made. The radian measure of an angle α (See Fig. 6) is determined by the following procedure:

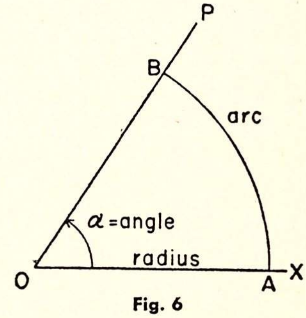

Fig. 6

With center at the vertex O of the angle α, and radius OA, draw an arc of a circle from the point, A, on the initial side, OX, to the point, B, on the terminal side; the number of radians in the angle, α, is then the length of the arc divided by the radius.

Using the word, *angle*, to denote the number of radians in the angle, α; the word, *radius*, to denote the length (say in inches) of the line segment, OA; and the word, *arc*, to denote the length (in the same

units) of the arc, AB, we can put this definition in the convenient form,

$$\text{angle} = \frac{\text{arc}}{\text{radius}}. \qquad \text{I}$$

It is clear that it makes no difference how large the radius, OA, is taken to be; for if OA is increased or decreased by any given factor, then the arc, AB, is increased or decreased by the same factor, and the quotient, arc divided by radius, remains the same as before. The reader should repeat "angle equals arc divided by radius" over and over to himself until the formula becomes one of his friends.

If the angle is one revolution, then the arc is one complete circumference of a circle and so is 2π times the radius; the angle is therefore equal to $2\pi r$ divided by r—that is, 2π radians. We thus obtain the formulas,

$$2\pi \text{ radians} = 1 \text{ revolution} \qquad \text{II}$$

and

$$2\pi \text{ radians} = 360°. \qquad \text{III}$$

The other measurements shown in Table XXXI (page 440) follow easily. These facts are used repeatedly in trigonometry and other subjects and should be memorized.

The value of π, correct to 15 decimal places, is

$$\pi = 3.141{,}592{,}653{,}589{,}793. \qquad \textbf{IVa}$$

The approximation,

$$\pi = 3.1416, \qquad \textbf{IVb}$$

is suffiicently accurate for most calculations.

The angle, α, shown in Fig. 6 contains one radian since the arc and the radius are equal. The number of degrees in one radian is computed from the formula,

$$1 \text{ radian} = \frac{180°}{\pi}.$$

The reader may use this formula to determine for himself the number of degrees, minutes, and seconds in one radian. Likewise, the formula,

$$1° = \left(\frac{\pi}{180}\right) \text{ radians},$$

may be used to determine the number of radians in 1 degree.

It has been stated that the radian is the most natural and useful unit for measurement of angles. One reason why this is so lies in the fact that the very simple formula,

$$\text{angle} = \frac{\text{arc}}{\text{radius}},$$

already referred to, holds when and only when the angle is measured in radians. This formula naturally enables us to compute the angle when the arc and the radius are known; but it also enables us to do more. It enables us to find any one of the three quantities, angle, arc, or radius, when the other two are known. Formula V,

$$\text{radius} = \frac{\text{arc}}{\text{angle}}, \qquad \text{V}$$

enables us to determine the radius when the arc and the angle are known, and Formula *VI*,

$$\text{arc} = \text{angle} \times \text{radius} \qquad \text{VI}$$

determines the arc when the radius and the angle are known.

One who makes routine use of these formulas would naturally memorize them; meanwhile it is sufficient to remember that "angle equals arc divided by radius" and to derive Formulas *V* and *VI* from Formula *I*.

When an angle (say θ) is measured in radians (say $\theta = 0.025$ radian), it is customary to omit the word, radian, and to say and write merely $\theta = 0.025$. Thus, when one says $\alpha = \frac{\pi}{2}$, one means that $\alpha = \frac{\pi}{2}$ radians, or 1 right angle, or 90°. If one writes $\alpha = 90$, one means that $\alpha = 90$ radians; if one wishes to say that α is 90 degrees, one writes $\alpha = 90°$.

A *milliradian* is one-thousandth of a radian, just as a millimeter is one-thousandth of a meter and a milligram is one-thousandth of a gram. (See page 41.) In cases where an angle is a small part of a radian (say 0.0075 radian), it may be regarded as a convenience to express the angle in milliradians.

For example, we can write $\alpha = 7.5$ milliradians instead of $\alpha = 0.0075$ radians, and Formulas *V* and *VI* can be written in the forms,

$$\text{radius} = \frac{1000 \times \text{arc}}{\text{milliradians}} \qquad \text{VII}$$

and

$$\text{milliradians} = \frac{1000 \times \text{arc}}{\text{radius}}. \qquad \text{VIII}$$

The number of milliradians in an angle, θ (θ radians), is 1000 θ.

Before attacking the following problems, you should test yourself by writing out (without the aid of textbook or notes) fundamental relations between units of an angle. (See Table XXXII, page 440.) If you do not have success, read the text and try again until you become successful. You should be prepared, at any place or time, to write formulas such as

$$1 \text{ right angle} = \frac{\pi}{2} \text{ radians.} \qquad \text{IX}$$

Illustrative Problem A

An enemy ship is known to be 300 feet long and 75 feet wide. From a point, O (see Fig. 7), you observe that the ship is steaming straight toward you and that the ship subtends an angle of 0.005 radians, or 5 milliradians. Find an estimate of the distance to the enemy ship.

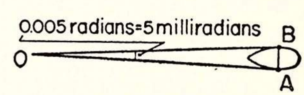

Fig. 7

Since the ship is seen to be miles away, it evidently makes no practical difference whether the width of the ship (75 feet) is measured along a straight line from A to B or along the arc, AB, of a circle with center at O. In fact, the arc, AB, is so nearly coincident with the line-

segment, AB, that one could not know whether the 75 feet was actually measured along the arc instead of along the line-segment.

Suppose the 75 feet to be the length of the arc. Then the computation,

$$\text{radius} = \frac{\text{arc}}{\text{angle}} = \frac{75}{0.005} = 15{,}000,$$

in which the angle is measured in radians, shows that the distance, OA, to the ship is 15,000 feet.

This computation may be put in the form,

$$\text{radius} = 1000 \times \frac{\text{arc}}{\text{milliradians}} = 1000 \cdot \frac{75}{5} = 15{,}000.$$

Illustrative Example B

Suppose that you are directing the operation of a gun and wish to hit an enemy gun 4,000 feet away. You fire a shot at the target, and receive a report that your shot landed 4,000 feet from your gun but 100 feet to the left of the target. (The 100 feet may be regarded as measured along the arc of a circle with center at your gun.) Through what angle would you turn your gun toward the right in order to make your next shot hit the target?

The number of radians in the angle, θ, is given by the formula,

$$\theta = \frac{\text{arc}}{\text{radius}} = \frac{100}{4{,}000} = 0.025.$$

The number of milliradians may be obtained from this result by the formula,

$$\text{milliradians} = 1000 \times \frac{\text{arc}}{\text{radius}} = 1000 \cdot \frac{100}{4{,}000} = 25.$$

Illustrative Example C

An arc (which may, for example, be a section of a railroad track) of a circle of radius 3400 feet is to be constructed so that it will subtend an angle of 29°30′ at the center of the circle. What is the length of the required arc?

Finding that $29°30' = 0.51486$ (that is, 0.51486 radians), we use Formula VI to obtain: $\text{arc} = 0.51486 \times 3400' = 1750.52'$.

Illustrative Example D

Engineers and physicists know that the *kinetic energy* of a wheel due to its angular velocity is equal to $\frac{1}{2}I\omega^2$ where I is the moment of inertia and ω is the angular velocity measured in radians per second. Find the kinetic energy of a wheel rotating 60 revolutions per second if $I = 3$.

Since the wheel rotates 60 revolutions each second, it rotates $60 \times 2\pi$, or 377 radians, each second. Hence, $\omega = 377$ and the kinetic energy is

$$\frac{1}{2} \times 3 \times 377^2 = 213{,}200.$$

Illustrative Example E

Suppose the moon travels in a circle of radius 240,000 with center at the earth, and that it travels through an angle of 13°10.5′ each day. (This supposition is nearly in accordance with the facts; the given angle does not take into account the "apparent" motion of the moon due to rotation of the earth.) Find, in miles per day and in miles per hour, the speed of the moon in its circular path.

$$13°10.5' = 0.22994 \text{ radians.}$$

From Formula VI, we find:

$$\begin{aligned} \text{arc} = 0.22994 \times 240{,}000 &= 55{,}200 \text{ miles per day} \\ &= 2{,}300 \text{ miles per hour.} \end{aligned}$$

TEST YOUR KNOWLEDGE OF ANGULAR MEASUREMENT

1 Verify some of the figures in Table XXXIII (page 440) which gives the radian measures of angles expressed in degrees and minutes. Show how you can obtain the number of radians in 23°46′ by adding four of the figures in the table.

2 A pulley is to be designed so that a belt traveling 40 feet per second will drive it (without slipping) at the rate of 25 revolutions per second. What should be the radius, in inches, of the pulley?

MILS

The remaining unit of angle which we have to introduce is the *mil.* The mil is defined by the formula,

$$1600 \text{ mils} = 1 \text{ right angle.}$$

The fundamental formulas which relate mils and radians are

$$1600 \text{ mils} = \frac{\pi}{2} \qquad \text{X}$$

or

$$1 \text{ mil} = \frac{\pi}{3200} = \frac{3.1416}{3200} \qquad \textbf{XI}$$

so that

$$1 \text{ mil} = 0.000982 \text{ radian} \qquad \textbf{XII}$$

and

$$1 \text{ mil} = 0.982 \text{ milliradian} \qquad \textbf{XIII}$$

It thus appears that, with an error of a little less than 2 per cent, a mil is the same as a milliradian. Hence, when errors of 2 per cent are not regarded as significant, a mil may be regarded as the same as a milliradian, that is, as one-thousandth of a radian. It is this fact that gives both name and usefulness to the mil.

If one assumes that a mil is a milliradian so that 1 radian = 1000 mils and the number of mils in an angle is 1000 times the number of radians, we can put the three relations involving angle, arc, and radius in the forms,

$$\text{mils} = \frac{1000 \text{ arc}}{\text{radius}}, \ \textbf{XIV} \qquad \text{arc} = \frac{\text{mils} \times \text{radius}}{1000}, \ \textbf{XV} \qquad \text{radius} = \frac{1000 \text{ arc}}{\text{mils}}. \ \textbf{XVI}$$

One who has many problems to solve would memorize all three formulas and would solve problems very quickly by means of them.

Illustrative Example A

A ship known to be 20,000 feet away from you is steaming directly away from you. The ship subtends an angle of 4 mils. How wide is the ship?

As $$\text{arc} = \frac{\text{mils} \times \text{radius}}{1000} = \frac{4 \times 20{,}000}{1000} = 80,$$

80 feet is (approximately) the width of the ship.

Illustrative Example B

You expect to pilot a ship between two buoys known to be 500 feet apart. You know that the buoys are equidistant from you, and that the angle subtended by the arc joining them is 24 mils. How far are you from the buoys?

As $$\text{radius} = \frac{1000 \text{ arc}}{\text{mils}} = \frac{1000 \times 500}{24} = 20{,}800,$$

the distance is approximately 20,800 feet.

One may wonder why anyone would use the mil as an approximation to the milliradian and thereby introduce systematic errors of 2 per cent into calculations. Why not use milliradians and eliminate the errors? It is doubtless true that the argument in favor of mils is the following: The number of mils in a right angle is a "nice" number (1600), whereas the number of milliradians in a right angle is an "ugly" number (about 1570.8). The question whether or not a given individual considers the nice number to be enough nicer than the ugly number to justify 2 per cent errors in calculations is a question which the individual must answer for himself.

TEST YOUR KNOWLEDGE OF MILS WITH THIS EXERCISE

3 Beginning with the definition of the mil, give (on paper) a full account of the procedure by which one arrives at the approximate formulas involving mils, arc, and radius.

Quadrants

The two coördinate axes, the X-axis and the Y-axis, separate the XY-plane into four *quadrants* (quarters), as we learned in discussing graphs (page 237). As indicated by the Roman numerals, I, II, III, and IV, in Fig. 8, these are called, respectively, the first, second, third, and fourth quadrants. The axes are not in the quadrants; they are the boundaries of the quadrants.

If $P(x, y)$ lies in the first quadrant, both x and y are positive. (Why?)
If $P(x, y)$ lies in the second quadrant, x is negative and y is positive. (Why?)

If $P(x, y)$ lies in the third quadrant, both x and y are negative. (Why?)
If $P(x, y)$ lies in the fourth quadrant, x is positive and y is negative. (Why?)
These facts will be used repeatedly.

Just as + and − signs are used to determine whether points are to the right or the left of the Y-axis and above or below the X-axis, so also + and − signs are used to determine whether an angle is generated by counterclockwise or clockwise rotation of a line.

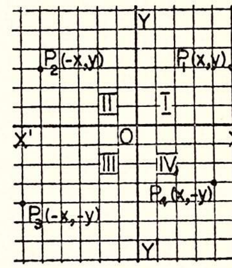

Fig. 8

For example, the angle, α, of Fig. 9 is generated by rotation of a line in a *positive* (counterclockwise) direction from the initial position, OX, to the terminal position, OA, and $\alpha=45°=\frac{\pi}{4}$.

The angle, β, of Fig. 9 is generated by rotation of a line in a *negative* (clockwise) direction from the initial position, OX, to the terminal position, OB, and $\beta=-45°=-\frac{\pi}{4}$.

The angle γ of Fig. 9 is obtained by rotation in a negative direction and $\gamma=-315°=-\frac{7\pi}{4}$.

As one sees from Fig. 9, the angles, α and γ, both have their terminal sides in the first quadrant, while the terminal side of β is in the fourth quadrant.

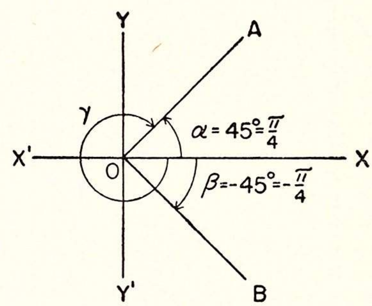

Fig. 9

TEST YOUR KNOWLEDGE OF QUADRANTS WITH THESE EXERCISES

Draw figures showing the following angles and show that the terminal sides lie in the quadrants indicated:

4 135°, II

5 −135°, III

6 1000°, IV

7 $\frac{5\pi}{6}$, II

8 9π, *between* the second and third quadrants

9 3, II

10 An angle, φ, is generated by rotating a line in the clockwise direction from the initial position, OX, until it first comes in contact with the point (1, −1). Show that $\varphi=-45°=-\frac{\pi}{4}$. (*Hint*: The figure which you should draw contains an isosceles right triangle, that is, a 45°-45°-90° triangle.)

A NOTE ON DIVISION

Before turning to the definitions of the trigonometric functions, we should make certain that we understand thoroughly a peculiarity of the number, 0. When we write (in trigonometry or anywhere else),

$$x=\frac{b}{a},$$

we mean that there is one and only one number, x, such that

$$ax=b$$

and that x is this number.

If $a \neq 0$, then for each b there is exactly one x such that $ax=b$, and this is $\frac{b}{a}$.

If $a=0$ and $b=0$, then the equation, $ax=b$, becomes $0x=0$; and since *every* number satisfies this equation, we conclude that $\frac{0}{0}$ is meaningless.

If $a=0$ and $b \neq 0$, then the equation, $ax=b$, becomes $0x=b$; and since *no* number x satisfies this equation, we conclude that $\frac{b}{0}$ is meaningless.

The conclusion we must emphasize is that the symbol, $\frac{b}{a}$, is undefined and completely meaningless when $a=0$.

As a final warning against division by 0, we may remark that the true equation,

$$0\times 2=0\times 3$$

does *not* imply that

$$2=3.$$

Zero is the only number by which we cannot divide. We note in particular that if r is a number not 0, then 0 divided by r is equal to 0.

THE TRIGONOMETRIC FUNCTIONS

In dealing with angles in trigonometry, we often find it convenient to refer to their *functions*. The six functions are:

sin θ (read sine theta),
cos θ (read cosine theta),
tan θ (read tangent theta),
cot θ (read cotangent theta),
sec θ (read secant theta),
csc θ (read cosecant theta).

The definitions must be thoroughly understood and memorized since, naturally, the definitions tell what the trigonometric functions are.

Let θ be any given angle (say 40° or 150° or $-136°$ or $\frac{\pi}{6}$ radians or -0.6237 radians).

Let a line start from the initial position, OX, and rotate in the proper direction (positive or negative) until it has generated the angle, θ, and let OP be the terminal side of the angle.

Let r denote the distance from the origin, O, to the point, P. The constructions are shown for four different angles in Fig. 10, on page 395. By taking $r=1$, we may obtain values for x and y expressed as decimals, or percentages of the unit value.

Definitions

The six trigonometric functions of θ are defined in terms of r and the coördinates of P by the formulas,

$$\sin\ \theta = \frac{y}{r}, \qquad \text{XVII}$$

$$\cos\ \theta = \frac{x}{r}, \qquad \text{XVIII}$$

$$\tan\ \theta = \frac{y}{x}, \qquad \text{XIX}$$

$$\cot\ \theta = \frac{x}{y}, \qquad \text{XX}$$

$$\sec\ \theta = \frac{r}{x}, \qquad \text{XXI}$$

$$\csc\ \theta = \frac{r}{y}, \qquad \text{XXII}$$

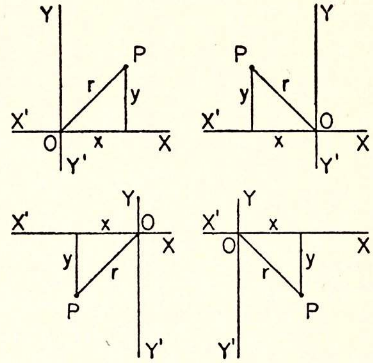

Fig. 10

except that, of course, if $x=0$, the quotients opposite tan θ and sec θ have zeros in the denominators and tan θ and sec θ are undefined; and if $y=0$, the quotients opposite cot θ and csc θ have zeros in the denominators and cot θ and csc θ are undefined.

Since r (being the distance from O to P) is always positive, the functions, sin θ and cos θ, are defined for all angles θ.

If θ is an angle for which the terminal side does not fall on one of the coördinate axes, then x, y, and r are all different from 0 and all six functions of θ exist.

In order to see that the six numbers, sin θ, etc., are determined by θ alone and are independent of the position of the point, P, on the terminal side of the angle, θ, it is only necessary to observe that, if P is moved away from or toward O along the line OP, then the three numbers, x, y, and r, will all be multiplied by the same positive number and the six quotients remain unaltered.

The remembering of the definitions is simplified by the fact that the first and last (sin θ and csc θ) are reciprocals, the second and fifth (cos θ and sec θ) are reciprocals, and the central pair (tan θ and cot θ) are reciprocals. Thus,

$$\cot\ \theta = \frac{1}{\tan\ \theta}, \quad \text{XXIII} \qquad \sec\ \theta = \frac{1}{\cos\ \theta}, \quad \text{XXIV} \qquad \csc\ \theta = \frac{1}{\sin\ \theta}. \quad \text{XXV}$$

The first three functions (sin θ, cos θ, and tan θ) are the ones most used; one may take the point of view that they are the most important functions and that the other three functions furnish convenient ways of writing their reciprocals.

Computation

The procedure for computing the trigonometric functions of a given angle may be summarized as follows: place the angle on the coördinate

system with the initial side coincident with OX; select a point, P, on the terminal side (any point except O itself); determine x, y, and r; and then compute the quotients to obtain the functions.

Notice that x and y are *coördinates* which may be either positive or negative depending upon the quadrant in which P lies; but that r (the *distance* from O to P) is always positive. We shall obtain familiarity with the trigonometric functions and their definitions by computing the values of functions of special angles.

TEST YOUR KNOWLEDGE OF TRIGONOMETRIC FUNCTIONS

11 Close your book and write the definitions of the trigonometric functions. Do this again and again until you can do it rapidly and accurately.

Functions of special angles

Some of the results to be obtained in this section are summarized in Table XXXIX (page 442).

FUNCTIONS OF 45°

Let us first find the functions of 45° ($\frac{\pi}{4}$ radians). The answers appear in the column under 45°.

The first step is to place the angle on the coördinate system as in Fig. 11.

The point, P, on the terminal side is so chosen that the x-coördinate of P is 1.

Since OQP is an isosceles right triangle in which

$$OQ = QP,$$

we see that the length of the side QP is 1.

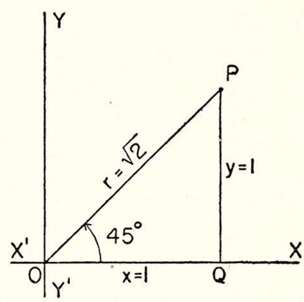

Fig. 11

Since P is *above* the X-axis, the y-coördinate of P is 1 (not -1).

The distance, OP, is the length of the hypotenuse of a right triangle having legs of lengths 1.

Hence, by the Pythagorean theorem (page 274),

$$OP = \sqrt{1^2 + 1^2} = \sqrt{2}.$$

Thus,

$$x = 1,$$
$$y = 1,$$
$$r = \sqrt{2}.$$

Therefore,

$$\sin 45° = \frac{y}{r} = \frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{2}}{2} = \frac{1}{2}\sqrt{2} = 0.707,$$

$$\tan 45° = \frac{y}{x} = \frac{1}{1} = 1,$$

and we have two of the functions of 45°.

The reader should obtain the other four and check with the table.

FUNCTIONS OF 30°

The first step in computing the functions of 30° and certain other angles is the construction of a 30°-60°-90° triangle.

This construction is accomplished by beginning with a complete equilateral triangle, OCB (Fig. 12).

By definition, each interior angle is a 60° angle.

The perpendicular bisector, OA, of the side, CB, bisects the angle at O and we obtain our 30°-60°-90° triangle in the upper part of Fig. 12.

We are free to let the length of a side, CB, of the equilateral triangle be any number we please; it is convenient to let

$$CB=2$$

so that AB (which is half of CB) will be the simple number, 1, involving no fractions.

This gives $AB=1$ and $OB=2$ as shown in Fig. 12, and the Pythagorean theorem gives $OA=\sqrt{3}$, as shown in the figure.

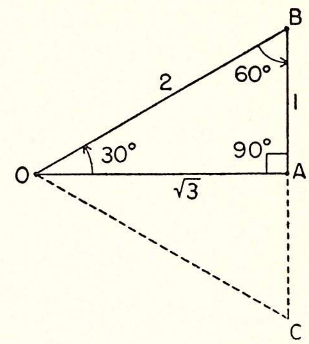

Fig. 12

TEST YOUR KNOWLEDGE OF VALUES OF FUNCTIONS

12 By the use of Fig. 12, show that Fig. 13 is the one you would draw when you want to find the functions of 30°. Verify that

$x=\sqrt{3}$, $y=1$, $r=2$, and $\sin 30°=\frac{y}{r}=\frac{1}{2}$.

Find the other functions of 30° and check with the table.

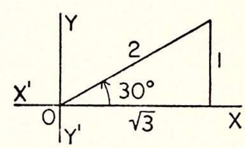

Fig. 13

13 By the use of Fig. 12, show that Fig. 14 is the one you would draw to find the functions of $-30°$. Be sure you see why the y-coördinate of P is -1 rather than $+1$. Show that $x=\sqrt{3}$, $y=-1$, $r=2$, and $\sin(-30°)=\frac{y}{r}=\frac{-1}{2}=-\frac{1}{2}$. Find the other functions of $-30°$ and check with the table.

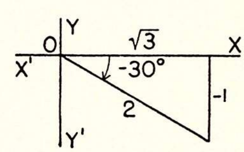

Fig. 14

14 Show that Figs. 15 and 16 are the ones you would draw to find the functions of 60° and 120°. Find the functions and check with the table.

Find all the functions of:

15 135° (90°+45°)

16 225° (180°+45°)

17 330° (270°+60° or 360°−30°)

18 If α is an angle with terminal side in the second quadrant and $\sin\alpha=\frac{3}{5}$, find all the functions of α.

19 If α is an angle with terminal side in the fourth quadrant and $\cos\alpha=\frac{\sqrt{3}}{2}$, find all functions.

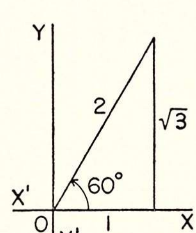

Fig. 15

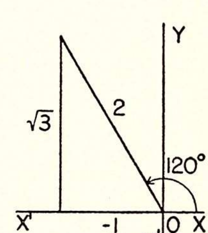

Fig. 16

FUNCTIONS OF 0°, 90°, 180°, 270°

We now apply the definitions of the trigonometric functions to find the functions of angles whose terminal sides lie on the boundaries between the quadrants. The results are given in Table XXXV (page 441). The dots opposite cot θ and under 0° indicate that cot 0° does not exist. The fact that some angles fail to have six trigonometric functions may be compared with the fact that some animals fail to have four feet; a zoölogist is not dismayed by the latter fact, and a student of trigonometry should not be dismayed by the former.

A figure showing the angle, 0° (or 0 radians), is especially simple. The initial and terminal sides of both coincide with OX, as in Fig. 17. If the point, P, on the terminal side is taken to be the point, (1, 0), then

$$x=1,$$
$$y=0,$$

and

$$r=1.$$

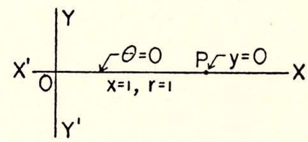

Fig. 17

Accordingly,

$$\sin 0=\frac{y}{r}=\frac{0}{1}=0,$$

$$\cos 0=\frac{x}{r}=\frac{1}{1}=1.$$

Similarly,

$$\tan 0=\frac{y}{x}=\frac{0}{1}=0.$$

Since $y=0$, the symbol $\frac{x}{y}$ is meaningless and therefore the angle, 0, has no cotangent.

Further consideration of ratios shows that

$$\sec 0=1,$$

and that 0 has no cosecant.

TEST YOUR ABILITY TO USE TABLES OF FUNCTIONS

20 Find the functions of 90°, 180°, and 270°, and check your results with those in Table XXXIX, page 442.

By drawing appropriate figures and either making or imagining that you make appropriate calculations, convince yourself that

21 sin θ is near 0 when θ is near 0.

22 sin θ is near 1 when θ is near 90°.

SIGNS OF THE TRIGONOMETRIC FUNCTIONS

We have seen that the trigonometric functions of θ are sometimes positive and sometimes negative.

If an angle, θ, has its terminal side in the first quadrant (quadrant I), then x, y, and r are all positive and all six functions are positive. This gives the first column of Table XXXIV (page 440).

If the terminal side of θ lies in the second quadrant, then x is negative while y and r are positive; hence, sin θ $\left(=\frac{y}{r}\right)$ and csc θ $\left(=\frac{r}{y}\right)$ are positive while the other four are negative. This gives the second column of the table.

TEST YOUR KNOWLEDGE OF SIGNS OF FUNCTIONS

23 Show that the last two columns of Table XXXIV are correct. (See page 440.)

24 With your book closed and without use of notes, construct the table which gives the algebraic signs of the six functions of angles with terminal sides in the four quadrants.

In what quadrant or quadrants must the terminal side of an angle θ lie if:

25 $\sin\theta$ is positive **26** $\cos\theta$ is negative **27** $\tan\theta$ is positive.

Work out your answers and check with the table of signs.

VALUES OF THE FUNCTIONS

One who has mastered the preceding sections of this chapter should find it very easy to obtain reasonably good estimates of the values

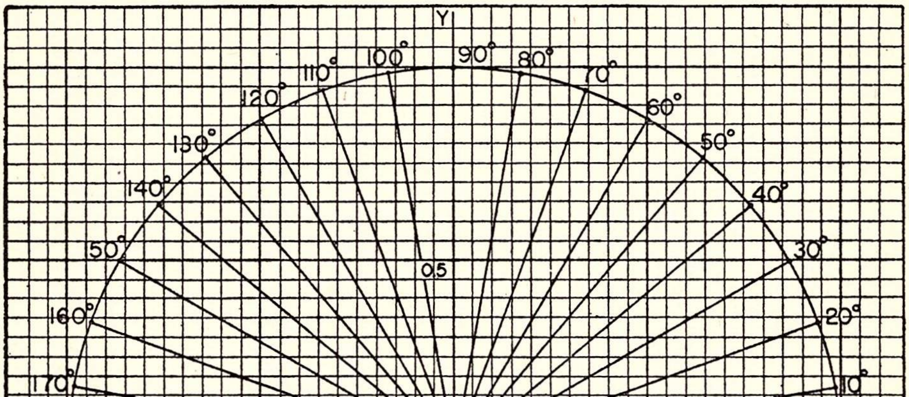

Fig. 18

of the trigonometric functions from Fig. 18.

The circle has a radius of 1, and the distance between two consecutive

rulings of the graph paper is 0.05. The numbers around the circle represent numbers of degrees in angles measured clockwise from the initial line, OX.

To obtain the functions of 30°, let the point opposite 30° be the point, P, for which

$$x=0.86\pm,$$
$$y=0.5\pm,$$

and

$$r=1.$$

Using these estimates, we obtain:

$$\sin 30°=0.5\pm,$$
$$\cos 30°=0.86\pm,$$

and

$$\tan 30°=0.58\pm.$$

The other functions are reciprocals of these.

More accurate values, obtained from page 443 in the tables are:

$$\sin 30°=0.50000,$$
$$\cos 30°=0.86603,$$

and

$$\tan 30°=0.57735.$$

The reader will see that, on page 443, the number, 0.50000, appears beside 30° and under *sin*; the number is, as we have seen otherwise, equal to sin 30°. The numbers, 0.86603 and 0.57735, appear beside 30° and under *cos* and *tan* respectively.

The reader should note that

$$\cos 31°45'=0.85035, \text{ etc.}$$

It is observed that the angles at the left sides of pages 442, 443, increase from 0° to 45°. To find a function of angles between 45° and 90° (say cos 72°15′), one first finds the angle, 72°15′, *above* the *72°* at the right side of page 443 and then finds the required number *above* the *cos* at the bottom of the page. Thus,

$$\cos 72°15'=0.30486,$$

and this checks very well with the more crude estimate obtained easily from Fig. 18.

TEST YOUR KNOWLEDGE OF VALUES OF FUNCTIONS

28 By use of Fig. 18, estimate different functions of different angles between 0° and 90° and check them by the more accurate figures in the tables. Include 45° in your list.

29 Find different functions of several different angles between 0° and 90° from the tables, and check them by estimates obtained from Fig. 18.

30 Show both by use of Fig. 18 and by use of the tables that, if θ is in the first quadrant and near 90°, then $\tan\theta$ is a large positive number. (More elaborate tables show that $\tan 89°59'=3437.7$ and that $\tan 89°59'50''=20{,}625.5$.)

Use Fig. 18 to obtain estimates of the sines, cosines, and tangents of:

31 −30° **32** −45° **33** 100° **34** 225°

35 It will be shown later that the sine of the sum of two angles, α and β, is given by the formula, $\sin(\alpha+\beta)=\sin\alpha\cos\beta+\cos\alpha\sin\beta$. Using the tables, check this formula for the values, $\alpha=15°$ and $\beta=20°$. Show that $\sin 35°$ is not equal to $\sin 15°+\sin 20°$.

36 If $\cos(\alpha-\beta)=\cos\alpha\cos\beta+\sin\alpha\sin\beta$, check for the values, $\alpha=20°$ and $\beta=10°$.

VARIATION AND GRAPHS OF THE FUNCTIONS

By use of Fig. 18, we see that sin θ

increases from 0 to 1 as θ increases from 0 to $\frac{1}{2}\pi$ (radians)

decreases from 1 to 0 as θ increases from $\frac{1}{2}\pi$ to π

decreases from 0 to -1 as θ increases from π to $\frac{3}{2}\pi$

increases from -1 to 0 as θ increases from $\frac{3}{2}\pi$ to 2π.

These facts show up clearly in Fig. 19, which exhibits the graph of sin θ.

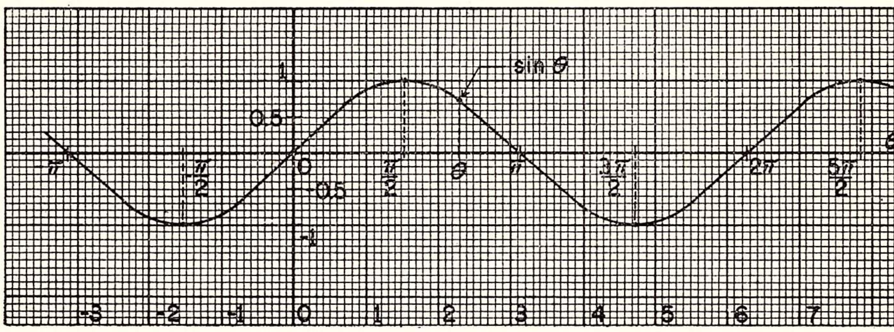

Fig. 19

For each θ, sin θ is *positive* when the curve is *above* the θ axis and is *negative* when the curve is *below* the θ axis.

The numerical value (or absolute value) of sin θ is the distance from the point, θ, to the curve, measured along a vertical line.

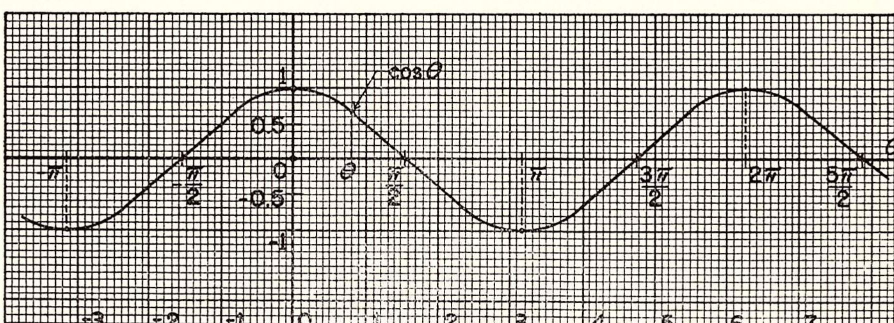

Fig. 20

The values of sin θ used in plotting the graph may be approximate values obtained by use of Fig. 18 or may be more accurate values obtained from the tables by use of the formulas (XXVI to XXVIII) of the next section.

A similar analysis of the variation and values of cos θ shows that the graph of cos θ is that shown in Fig. 20. After looking at Figs. 19 and 20, one may guess that, if the graph of sin θ in Fig. 19 were shifted to the left $\frac{\pi}{2}$ (1.57) units, it would become a graph of cos θ.
That this guess is correct is a consequence of the formula,

$$\sin\left(\frac{1}{2}\pi+\theta\right)=\cos\theta;$$

which will be introduced in the next section.

TEST YOUR KNOWLEDGE OF VALUES OF FUNCTIONS

37 By use of Fig. 18, and any values of the functions you know or select from tables, plot graphs of tan θ, cot θ, sec θ, and csc θ. You will find that the graph of tan θ does not intersect the line of points for which $\theta=\frac{\pi}{2}$, and that the graph runs off of your paper near $\theta=\frac{\pi}{2}$.

REDUCTION FORMULAS

The tables do not give functions of angles outside the range from 0° to 90° $\left(0 \text{ to } \frac{\pi}{2}\right)$. It is expected that you learn to use reduction formulas by means of which functions of angles not between 0° and 90° are given in terms of functions of angles between 0° and 90°.

For example, after one has learned that $\cos(90°+\theta)=-\sin\theta$, one can find cos 123° by setting $\theta=33°$. Thus, $\cos 123°=\cos(90°+33°)=-\sin 33°=-0.54464$.

Similarly, by use of the formula, $\sin(-\theta)=-\sin\theta$, one finds that

$$\sin\left(-\frac{\pi}{6}\right)=-\sin\frac{\pi}{6}=-\sin 30°=-\frac{1}{2}=-0.50000.$$

The following discussion of the necessary reduction formulas is phrased in terms of degree measure; for many purposes, one replaces 90° by $\frac{\pi}{2}$, 180° by π, etc., and considers θ to be measured in radians.

The formulas,

$$\sin(360°+\theta)=\sin\theta,\quad \cos(360°+\theta)=\cos\theta,\quad \text{and}\quad \tan(360°+\theta)=\tan\theta,$$

are obviously correct: a point, P, on the terminal side of the angle, θ, is at the same time a point, P, on the terminal side of the angle, $(360°+\theta)$; hence, the functions of the angles, θ and $(360°+\theta)$, are equal to the quotients of the same numbers. In the same way, it is seen that, if n is an integer (1, -1, 2, -2, etc.), then

$$\sin(360°n+\theta)=\sin\theta, \text{ etc.}$$

These formulas convert functions of all angles into functions of angles in the interval from 0° to 360°.

For example,

$$\begin{aligned} &\text{since} \quad 720 = 2\times 360, \\ &\sin \quad 1000° = \sin\ (720°+280°) = \sin 280°, \\ &\cos \quad -300° = \cos\ (-360°+60°) = \cos 60°. \end{aligned}$$

Table XXXVI (which also has other uses) may be used to convert functions of angles from 0° to 360° into functions of angles in the interval from 0° to 90° so that we can use the tables to find the functions.

These tables furnish 24 reduction formulas of the type,

$$\sin(-\theta) = -\sin\theta,\ \cos(180°-\theta) = -\cos\theta$$
$$\cos(90°+\theta) = -\sin\theta,\ \tan(90°-\theta) = \cot\theta.$$

We shall not prove these formulas, but we look briefly at a special case to see how the proofs would run.

If θ is a first-quadrant angle and P is a point on the terminal side of the angle, θ, with coördinates, (A, B), as in Fig. 21, then (as one sees by use of similar triangles) the point, Q, with coördinates, $(-B, A)$, is a point on the terminal side of the angle $(90°+\theta)$.

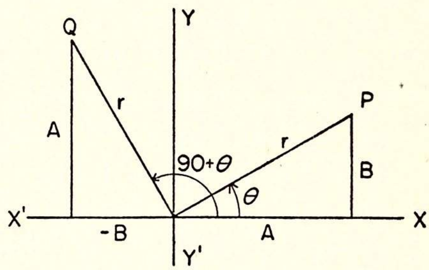

Fig. 21

Since

$$OP = OQ = r,$$

we find that

$$\cos(90°+\theta) = \frac{-B}{r} = -\frac{B}{r} = -\sin\theta.$$

Proofs for other functions and for angles in other quadrants are similar.

One who wishes to learn the reduction formulas should, instead, learn the following rules for writing them correctly:

The *cofunction of the sine* is the cosine, and the *cofunction of the cosine* is the sine.

Similarly, the tangent and cotangent are *cofunctions*.

A look at the reduction formulas shows that, if *function* stands for any one of the functions (sine, cosine, or tangent), then

$$\text{function } (90°n \pm \theta) = \pm\text{function } \theta \qquad \textbf{XXVI}$$

when n is even (0 or 2); and

$$\text{function } (90°n \pm \theta) = \pm\text{cofunction } \theta \qquad \textbf{XXVII}$$

when n is odd (1 or 3). The tables give the correct algebraic sign in each case. There is an easy way to figure out the sign so that one does not need to remember it.

By the above rule,

$$\cos(90°+\theta) = ?\ \sin\theta,$$

where ? is to be replaced by + or −. If θ is a first-quadrant angle $(0° < \theta < 90°)$, then $\sin\theta$ is positive and $\cos(90+\theta)$ is negative so ? could not be +; it must be −. The rule is this: Determine the sign so that the sides of the equality will be both positive (or both negative) when θ is a first-quadrant angle.

TEST YOUR KNOWLEDGE OF REDUCING FUNCTIONS

Use the two rules to express the following as functions of θ:

38 $\sin(90°-\theta)$ **41** $\sin(180°-\theta)$ **44** $\sin(270°+\theta)$ **46** $\cos(-\theta)$
39 $\cos(90°+\theta)$ **42** $\cos(180°+\theta)$ **45** $\sin(-\theta)$ **47** $\tan(-\theta)$
40 $\tan(90°+\theta)$ **43** $\tan(270°-\theta)$

In each case, check with the table of reduction formulas. Show, both by use of reduction formulas and by checking with the tables, that

48 $\sin 20° = \cos 70°$ **50** $\cos 44° = \sin 46°$
49 $\sin 32°15' = \cos 57°45'$ **51** $\tan 30° = \cot 60°$

Check the computations

52 $\sin 200° = \sin(180°+20°) = -\sin 20° = -0.34202.$
53 $\sin 200° = \sin(270°-70°) = -\cos 70° = -0.34202.$

Find each of the following in two ways:

54 $\sin 123°$ **55** $\cos 234°$ **56** $\tan 345°$

Make your results check with each other and with approximations obtained from Fig. 18.

Finding angles when functions are known

Trigonometric tables (See pages 442, 443) give the values of the trigonometric functions of some of the angles spaced between 0° and 90°. These tables enable us to find the angles (actually, approximations to the angles) for which a given function of the angle has a given value.

Illustrative Example A

Suppose we know that θ is an acute angle

$$(0<\theta<90°)$$

and that

$$\sin\theta = 0.28820.$$

The number, 0.28820, is found on page 442 of the table under the heading, *sin* and we see that

$$\theta = 16°45'.$$

If we know that φ is an acute angle and $\sin\varphi = 0.40000$, we look in the table and find that 0.40000 does not appear in the table of sines. However, we see that

$$\begin{aligned} \sin 23°30' &= 0.39875, \\ \sin\varphi &= 0.40000, \\ \sin 23°45' &= 0.40275, \end{aligned}$$

and this indicates that φ is *between* 23°30' and 23°45'.

If one is not disturbed by errors less than 15', one notices that 0.40000 is nearer to 0.39875 than to 0.40275 and one says that

$$\varphi = 23°30'.$$

It is easy to obtain a better value for φ by *interpolation*.

The number, 0.40000, lies $\frac{175}{400}$ of the way from 0.39875 to 0.40275; hence, we take, for our approximation to φ, an angle $\frac{175}{400}$ of the way from 23°30' to 23°45'—that is (to nearest minute),

$$\varphi = 20°35'.$$

Illustrative Example B

Find, to the nearest minute, the angle α if α is an acute angle and $\tan\alpha=\frac{2}{7}$.

We find that

$$\begin{aligned}\tan 15°45' &= 0.28203\\ \tan\alpha &= 0.28570\\ \tan 16°0' &= 0.28675\end{aligned}$$

and determination of the angle $\frac{367}{472}$ of the way from 15°45′ to 16° gives

$$\alpha=15°57'.$$

Illustrative Example C

If α and β are acute angles, $\sin\alpha=\frac{1}{3}$, and $\cos\beta=\frac{3}{5}$, find α, β, $\alpha+\beta$, $\cos\alpha$, $\sin\beta$, and $\cos(\alpha+\beta)$.

You can check your results by substituting in the formula,

$$\cos(\alpha+\beta)=\cos\alpha\cos\beta-\sin\alpha\sin\beta. \qquad \text{XXVIII}$$

SELECTIVITY IN USING FUNCTIONS

Since, as we know,

$$\sin 30°=\frac{1}{2}$$

and

$$\sin 150°=\sin(180°-30°)=\sin 30°=\frac{1}{2},$$

it follows that the angles, 30° and 150° (and all angles obtained by adding or subtracting integer multiples of 360° to or from these), have sines equal to $\frac{1}{2}$.

It can be shown that these are the only angles for which the sine is equal to $\frac{1}{2}$. Therefore, if you find that θ is an angle for which $\sin\theta=\frac{1}{2}$, then you know that θ must be one of the angles, 30°, 150°, 30°+360°, 150°+360°, 30°−360°, . . .; but you cannot say *which* of these is θ unless you have information in addition to $\sin\theta=\frac{1}{2}$.

If you know that $\sin\theta=\frac{1}{2}$ and that θ is an interior angle of a triangle (so that, as you learned in plane geometry) θ must be between 0° and 180°, then you know that θ must be either 30° or 150°.

Since you can draw a triangle (or some other figure) in which an interior

angle is 30° and one in which the angle is 150°, you are unable to say (unless you have further information) whether $\theta = 30°$ or $\theta = 150°$.

On the other hand, if you know that φ is an interior angle of a triangle (so that φ is between 0° and 180°) and if you know that $\cos\varphi = \frac{1}{2}$, then you can say that $\varphi = 60°$, for 60° is the only acute angle for which the cosine is $\frac{1}{2}$; and if φ were in the second quadrant, then $\cos\varphi$ could not be $\frac{1}{2}$ because $\cos\varphi$ would then be negative. It is easy to prove that, if θ is an interior angle of a triangle and $\cos\theta$ is known, then θ can be determined without ambiguity.

Illustrative Example

Find, to the nearest degree, the angle α if α is an interior angle of a triangle and $\cos\alpha = -\frac{3}{5}$.

Since

$$0° < \alpha < 180°$$

and $\cos\alpha$ is negative, α must be a second-quadrant angle.

Hence, $180° - \alpha$ is an acute angle and

$$\cos(180° - \alpha) = -\cos\alpha = \frac{3}{5} = 0.60000.$$

$$180° - \alpha = 53°8'$$

and

$$\alpha = 126°52'.$$

TEST YOUR KNOWLEDGE OF ANGULAR FUNCTIONS

Find all values of θ between 0° and 360° for which

57 $\sin\theta = 0.70711$ **59** $\tan\theta = -2.05030$ **61** $\cos\theta = 0.45010$

58 $\cos\theta = -0.25038$ **60** $\sin\theta = 0.25038$

Check your results against approximations obtained from Fig. 18.

PROBLEMS INVOLVING RIGHT TRIANGLES

This section is concerned with problems solvable by means of right triangles. All angles of which the trigonometric functions are used in this chapter are acute angles of right triangles. The number of degrees in each such angle is a positive number less than 90; hence, the six functions of these angles will all be positive numbers which can be determined directly from the tables.

A right triangle was defined in plane geometry (page 320) as a triangle in which one of the angles (the angle at R in Fig. 22) is a right angle. The other two angles are acute; let θ be one of the acute angles. It is convenient to name the sides of the triangle as an observer at the vertex, O, of the angle, θ, would name them:

The side, OP, opposite the right angle is, as in plane geometry, called the *hypotenuse*.

The side, RP, opposite the angle, is called the *opposite side*.

The side, OR, adjacent to the angle, is called the *adjacent side*.

According to the definitions of the trigonometric functions, we find sin θ, etc., by making O the origin and OR the X-axis of an xy-coördinate system, determining the coördinates, (x, y), of the point, P, on the terminal side of the angle and the distance r, (OP), and then computing the required quotients.

Functions of acute angles

When this is done, one sees that x, y, and r are all positive (since $0° < \theta < 90°$) and that

$$x = \text{adjacent side}$$
$$y = \text{opposite side}$$
$$r = \text{hypotenuse}$$

where the words on the right stand for numbers representing the lengths of the sides of the triangle. This enables us to write the first three functions of an acute angle of a right triangle in the form,

$$\sin\theta = \frac{\text{opposite side}}{\text{hypotenuse}}, \qquad \text{XXIX}$$

$$\cos\theta = \frac{\text{adjacent side}}{\text{hypotenuse}}, \qquad \text{XXX}$$

$$\tan\theta = \frac{\text{opposite side}}{\text{adjacent side}}. \qquad \text{XXXI}$$

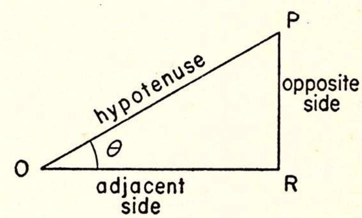

Fig. 22

These formulas should be remembered in this form; the formulas for cot θ, sec θ, and csc θ are best remembered by keeping in mind Formulas XXIII to XXV (page 395).

The advantage of these formulas (which apply *only* to acute angles) lies in the fact that the six functions of an acute angle of a right triangle can be written immediately in terms of the lengths of the sides; it is not necessary to place the angle upon a coördinate system to obtain the relations. We shall use these formulas repeatedly.

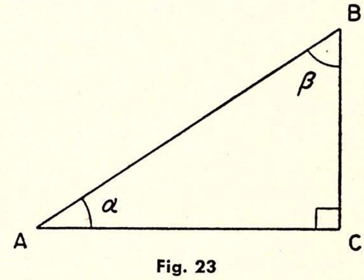

Fig. 23

The two acute angles, say α and β, of a right triangle are *complementary*, that is,

$$\alpha + \beta = 90°$$

or

$$\beta = 90° - \alpha.$$

Using Formula XXVI, we see that

$$\sin\beta = \sin(90° - \alpha) = \cos\alpha.$$

This result is obtained otherwise and very easily from Fig. 23, since the side opposite β is the side adjacent to α.

Thus,
$$\sin\beta = \frac{AC}{AB} = \cos\alpha.$$

Similarly, $\cos \beta = \sin \alpha$
and $\tan \beta = \cot \alpha$.
Each function of β is its cofunction of α.

ANGLES OF ELEVATION AND DEPRESSION

The terms, *angle of elevation* and *angle of depression*, are often used; we now define the terms.

Let E (perhaps your eye, or the location of an instrument for measuring angles) and A be points in a vertical plane, and let A lie above the horizontal line, EH, in this plane.

The angle, HEA (α in Fig. 24), is then the *angle of elevation* of the point, A, as viewed from E.

Similarly, when B is in the vertical plane and below the horizontal line, EH, the angle, HEB (β in Fig. 24), is the *angle of depression* of the point, B, as viewed from E.

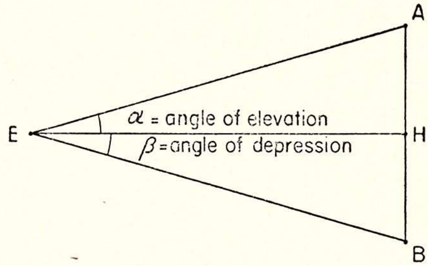

Fig. 24

TEST YOUR KNOWLEDGE OF ACUTE ANGLES

62 Without aid of book or notes, write out formulas giving the functions of an acute angle of a right triangle in terms of the sides.

Looking at Fig. 24, write the sines, cosines, and tangents of α and β in terms of the sides of the triangles.

Solving right triangles

When solving a problem involving triangles, one should *draw* a figure (even though the figure may be in the text) and use the figure to discover a way to solve the problem. The arithmetic computations occurring in this chapter and the next can be done most rapidly by one accustomed to use of a slide rule. More accurate computations are made most efficiently on computing machines. All standard textbooks on trigonometry emphasize the use of logarithms. When one finds himself professionally engaged in computational work, one should master all of the aids to computation.

Illustrative Example A

Suppose we want to find the height of a tall tree. We measure (Fig. 25) a distance out on a horizontal line from the base of the tree (say, 80 feet).

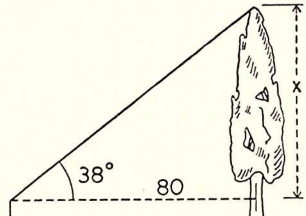

Fig. 25

We measure the angle of elevation of the top of the tree from a point 4 feet above the end of this line (say, the angle is 38°).

As indicated in Fig. 25, let x denote the height of the part of the tree above the 4-foot level.

Then $$\tan 38° = \frac{x}{80}$$
so that
$$x = 80 \cdot \tan 38° = 80 \times 0.78129 = 62.5 \text{ ft.}$$
The height of the tree is, therefore, 62.5 ft.

Illustrative Example B

Suppose we wish to find the distance from a point, O (our gun), to a point, E (enemy gun), and that the laying of a tapeline from O to E is impractical.

Suppose a point, J (Fig. 26), 700 yards east of O is found to be south of E, and that measurement (by a transit) shows that $\angle JOE = 65°15'$.

The distance, x, from O to E can then be computed from the formula,
$$\cos 65°15' = \frac{700}{x}.$$

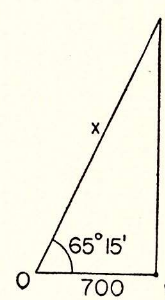

Fig. 26

We find that
$$x = \frac{700}{\cos 65°15'} = 700 \sec 65°15' = 1672 \text{ yards.}$$

Illustrative Example C

Show that a triangle of which the sides have lengths 3, 4, and 5 is a right triangle, and find the smallest angle.

The triangle is a right triangle because $3^2 + 4^2 = 5^2$.

The smallest angle (call it α) is opposite the smallest side and by use of a figure we see that $\sin \alpha = 0.60000$. Hence, $\alpha = 36°52'$.

Illustrative Example D

A vertical yardstick casts a shadow 41 inches long. Find the angle of elevation of the sun.

The tangent of the angle is $\frac{36}{41} = 0.87800$ and the angle is $41°17'$.

Illustrative Example E

Many problems, phrased in very different terms, have solutions similar to the solution of the following problem.

Suppose we are required to find the height, x, of a point, D, on a cliff above a horizontal base line, AB (Fig. 27).

The point, C, of the figure is imbedded in the rocks, and we wish to use only measurements which can be made without digging.

Suppose the distance, a, and the angles, α and β, are known.

Our problem is to determine x in terms of them.

There are times in trigonometry (as well as in algebra) when the solution of a problem is simplified by the introduction of more than one "unknown". Let y denote the distance from B to C.

Then, using the angles, α and β, we may write immediately the two equations,

$$\tan\beta=\frac{x}{y}, \qquad \text{a}$$

$$\tan\alpha=\frac{x}{a+y}. \qquad \text{b}$$

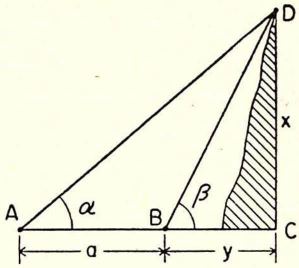

Fig. 27

Multiply a by y and divide by $\tan\beta$.
Multiply a by $(a+y)$ and divide by $\tan\alpha$.
This gives

$$y=x\cot\beta, \qquad \text{c}$$

$$y+a=x\cot\alpha. \qquad \text{d}$$

Subtracting a from b eliminates y and gives

$$a=x(\cot\alpha-\cot\beta) \qquad \text{e}$$

so that

$$x=\frac{a}{\cot\alpha-\cot\beta} \qquad \text{f}$$

This is our result: a formula which gives the height, x, of the cliff in terms of the measured quantities, a, α, and β.

Suppose $a=120$ feet, $\alpha=27°$, and $\beta=53°$.

Then $x=\dfrac{120}{\cot 27°-\cot 53°}=\dfrac{120}{1.96261-0.75356}=\dfrac{120}{1.20905}=99$ feet.

Using Fig. 27, obtain the formula,

$$x=\frac{a}{\cot\alpha-\cot\beta},$$

by beginning with the use of $\cot\alpha$ and $\cot\beta$. (This is quicker than beginning with $\tan\alpha$ and $\tan\beta$.)

Illustrative Example F

Two observers stationed 8000 feet apart report that an airplane is directly above the line joining them and that the two angles of elevation are 38° and 28°. How high is the airplane?

Let α and β be the two angles, h be the height of the airplane, d be the distance between the observers, and x and $d-x$ be the distances shown in Fig. 30. Then

$$\frac{x}{h}=\cot\alpha, \qquad \text{a}$$

$$\frac{d-x}{h}=\cot\beta, \qquad \text{b}$$

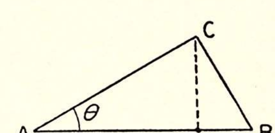

Fig. 28

and addition of a and b eliminates x to give

$$\frac{d}{h}=\cot\alpha+\cot\beta \qquad \text{c}$$

and hence

$$h=\frac{d}{\cot\alpha+\cot\beta}. \qquad \text{d}$$

Setting $d=8{,}000$, $\alpha=38°$, and $\beta=28°$ gives

$$h=\frac{8{,}000}{1.27994+1.88073}=\frac{8{,}000}{3.16067}=2530 \text{ feet}$$

as the answer to the problem.

TEST YOUR KNOWLEDGE OF RIGHT TRIANGLES

63 From a point 80 feet out on a horizontal line from the base of a vertical tower, the angle of elevation of the top of the tower is 49°. What is the height of the tower?

64 Show that the area of the triangle in Fig. 28 is given by the formula, area$=\frac{1}{2}\,AB\cdot AC\sin\theta$. (*Hint*: Notice that $\sin\theta=\frac{DC}{AC}$ and hence that $DC=AC\cdot\sin\theta$.)

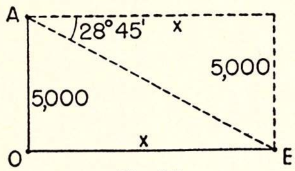

Fig. 29

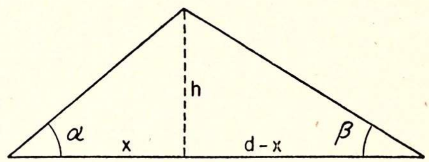

Fig. 30

65 From an airplane, A, 5,000 feet above a ship, O, it is observed that the angle of depression of a ship, E, is 28°45′. (See Fig. 29.) How far apart are the ships?

66 A regular polygon with 60 sides is inscribed in a circle with radius of 10. Find its perimeter. (*Hint*: The perimeter is 120 times the length of half of one side. You should expect your answer to be nearly 20π, the circumference of the circle.)

PROBLEMS INVOLVING OBLIQUE TRIANGLES

An *oblique triangle* is defined as a triangle in which no one of the angles is a right angle. The formulas of this chapter apply to right triangles as well as to oblique triangles. One who understands the discussions and solves the problems in this chapter should be able to solve many practical problems involving triangles. He should also be able to solve many problems involving more complicated figures by cutting the more complicated figures into triangles and working with the triangles.

The law of sines

Suppose we know a side of a triangle and the angles at the vertices at the ends of the side. (Remember that if two angles are given, we immediately compute the third and know all three.)

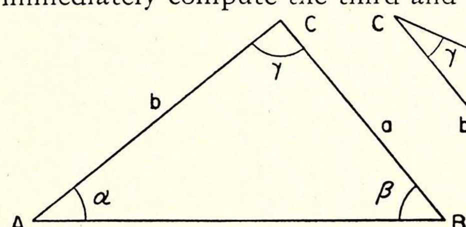

Fig. 31

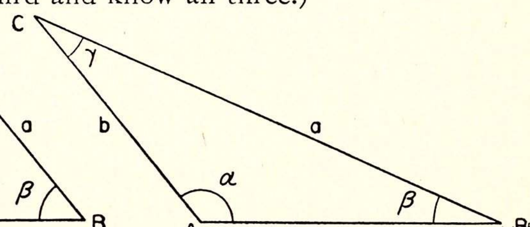

Fig. 32

Let the known side be AB, and call its length c, as in Fig. 31 or Fig. 32.

Draw lines above AB, the line AC, forming the known angle, α, at A, and line, BC, forming the known angle, β, at B.

The intersection, C, of the two lines is the third vertex, C. This completes the construction of the triangle.

How shall we compute a and b, the unknown lengths of two sides of the triangle? We can do it by use of the *law of sines*, which states that, if α, β, and γ are the three angles of a triangle and if a, b, and c are (in order) the lengths of the sides opposite these angles, then

$$\frac{a}{\sin\alpha}=\frac{b}{\sin\beta}=\frac{c}{\sin\gamma}. \qquad \text{XXXII}$$

There are different ways of proving the law of sines. The following way is interesting because it shows more than that the three quotients in XXXII are equal to each other; it shows that each of them is equal to the diameter of the circle circumscribing the triangle (Fig. 33).

Commencing with the triangle, ABC (Fig. 33), we construct the circumscribing triangle (with center at O); draw the diameter, COD, from C; and draw the line, DB.

$$\angle CDB=\alpha,$$

$$\angle CAB=\angle CDB=\frac{1}{2}\angle COB,$$

$$\angle CDB=90^\circ \text{ (inscribed in a semi-circle)},$$

$$\therefore\quad \triangle CDB=\text{rt. }\triangle\,\cdot$$

and $$\sin\alpha=\frac{a}{2R}, \qquad \text{XXXIII}$$

where R is the radius of the circumscribed circle, and $2R$ is the diameter. This shows that

$$\frac{a}{\sin\alpha}=2R. \qquad \text{XXXIV}$$

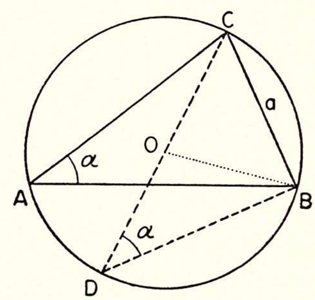

Fig. 33

When a triangle is given, we can let α represent any one of the three angles we please, and let a be the length of the oppòsite side.

Hence, XXXIV says that, in any triangle, the length of a side divided by the sine of the opposite angle is equal to the diameter of the circumscribed circle.

Hence, b divided by $\sin\beta$ is $2R$, and c divided by $\sin\gamma$ is $2R$. Therefore,

$$\frac{a}{\sin\alpha}=\frac{b}{\sin\beta}=\frac{c}{\sin\gamma}=2R. \qquad \text{XXXV}$$

TEST YOUR KNOWLEDGE OF THE SINE LAW

67 Using Fig. 34, show that

$$\sin\alpha=\frac{h}{b},$$

$$\sin\beta=\frac{h}{a},$$

and use these formulas to show that

$$\frac{a}{\sin\alpha}=\frac{b}{\sin\beta}.$$

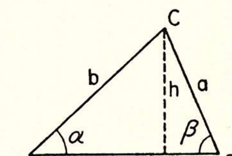

Fig. 34

68 Show, by study of appropriate figures, that these formulas hold when (a) α is a right angle and (b) α is an obtuse angle (that is, $90° < \alpha < 180°$). The idea used here provides another proof of the law of sines.

USING THE SINE LAW

The law of sines can be used when three of the four quantities (two sides and the two angles opposite them) are known, for, if the sides and angles have measures, a, b, α, and β, then

$$\frac{a}{\sin \alpha} = \frac{b}{\sin \beta}.$$

If both angles and one side are known, then by use of a little algebra and of trigonometric tables we can find the other side. If both sides and one angle (say, α) are known, then algebraic manipulation gives

$$\sin \beta = \frac{b}{a} \sin \alpha, \qquad \text{XXXVI}$$

so that $\sin \beta$ may be computed.

If it turns out that $\sin \beta = 1$, then we know that $\beta = 90°$, but if it turns out that $\sin \beta$ lies between 0 and 1, say $\sin \beta = 0.8829$, then we know that β is either the acute angle, 62°, or the obtuse angle, 118°; but we do not know which.

In fact, as shown in Fig. 35, there are two triangles for which α, a, and b have the same values; β is acute in one and obtuse in the other.

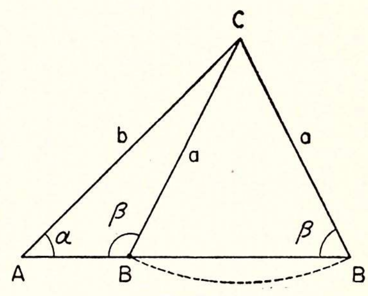

Fig. 35

To give two sides and the angle opposite one of them and ask for the other angles and sides of *the* triangle is like asking for the height and weight of *the* senator from Ohio. There is no "*the triangle*" and no "*the senator*"; there are two triangles and two senators. It is, however, sensible to ask for the shapes (plural) of the triangles (plural); and it is sensible to ask for the other sides and angles of the triangle for which β is acute.

The important applications of the law of sines come when at least one side and two (and hence, after a little arithmetic, all three) angles of the triangle are known.

It is difficult to measure the distance between two points accurately when one of the points is in the air or on the sea, and it is frequently difficult when both points are on land because of rough ground, etc., but it is easy to measure angles accurately. Hence, problems of the following type are good practice problems; the law of sines is important because it enables us to solve them easily.

Illustrative Example

Suppose observers at A and B, 4000 yards apart (Fig. 36) report that a cruiser is at the point, C, and that $\alpha=49°$, $\beta=74°$. What is the distance b from A to C?

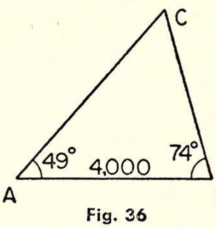

Fig. 36

We first find that $\alpha+\beta=123°$ and $\gamma=180°-123°=57°$.

Hence, use of the formula, $\frac{b}{\sin\beta}=\frac{c}{\sin\gamma}$, gives

$$b=\frac{c\sin\beta}{\sin\gamma}=\frac{4000\sin 74°}{\sin 57°}=\frac{4000\times 0.96126}{0.83867}=4582+\text{yards.}$$

TEST YOUR ABILITY TO APPLY THE SINE LAW

69 Draw a figure similar to Fig. 36, but with the 4000 deleted and with the AC-line as 4582. Then compute the length, c, of the side, AB. (If you do not suspect that your answer should turn out to be about 4000, you are missing something.)

70 Formulate practical problems in which two angles and one side of a triangle are assigned numerical values and one or more of the remaining sides and angles are to be found. Draw the figures carefully on graph paper and estimate answers. Compute the answers by use of the law of sines. Make your graphs and computations check each other. (*Remark:* If you have not used the idea before, notice that, since $\tan 38°30'=0.79544$, you can draw an excellent picture of a $38°30'$ angle by constructing on graph paper a right triangle in which one side is 10, or 100, units long and the other is 7.7, or 77.6, units long.)

71 The angles of a triangle are 15°, 35°, and 130°, and the shortest side is 18 inches long. What is the radius of the circumscribed circle?

$$\left(r=\frac{abc}{4\sqrt{s(s-a)(s-b)(s-c)}}\text{ where } s=\frac{a+b+c}{2}\right)$$

The law of cosines

Suppose that two sides (say, b and c) and the angle, α, between them are known.

The complete triangle is easily constructed: one first constructs the angle, α, with vertex at A; then marks points B and C on the sides of the angle at distances, c and b, from A; and finally draws the line-segment, BC, to obtain the triangle, ABC.

How can we compute the length, a, of the side, BC?

A method is provided by the *law of cosines*, which states that, if b and c are the lengths of the sides of a triangle and α is the angle included between b and c, then the length, a, of the third side (the side opposite α) is given by

$$a^2=b^2+c^2-2bc\cdot\cos\alpha. \qquad \textbf{XXXVII}$$

Of course, a is the square root of a^2; that is, the positive number whose square is a^2.

The most frequent applications of the law of cosines occur in problems in physics and engineering when two line-segments (representing

beams, forces, velocities, etc.) have known lengths (say, b and c) and intersect at a known angle (say, α); one is required to compute the length, a, of the line joining the ends of the line-segments, and uses the law of cosines for this purpose. As one sees by looking at XXXVII, the law of cosines also enables us to compute angles; for if a, b, and c are known, then XXXVII enables us to determine first cos α and then α.

To prove the law of cosines, let the triangle be placed in an XY-plane with A at the origin, with B on the X-axis to the right of the origin, and with the vertex, C, above the X-axis; and let the coördinates of C be (x, y).

Fig. 37 portrays the result for a case in which α and β are both acute angles.

If α were an obtuse angle, C would be to the left of the Y-axis and the coordinate, x, would be negative.

If the angle, β, at B were obtuse, C would be to the right of a vertical line through B and x would be greater than c.

If α is a right angle, x is 0; and if β is a right angle, $x=c$. In each case, applications of the Pythagorean theorem give

$$\begin{aligned} a^2 &= y^2+(c-x)^2 \\ &= b^2-x^2+(c-x)^2 \\ &= b^2+c^2-2cx. \end{aligned}$$

But

$$\frac{x}{b}=\cos\alpha$$

so that

$$x=b\cdot\cos\alpha.$$

Therefore,

$$a^2=b^2+c^2-2bc\cdot\cos\alpha, \qquad \text{XXXVIII}$$

and the law of cosines is proved.

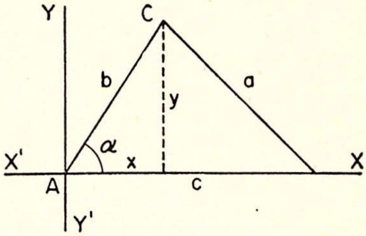

Fig. 37

This law may be stated in words as follows: in any triangle, the square of one side is equal to the sum of the squares of the other sides minus twice the product of the other two sides and the cosine of the angle opposite the original side. Hence, it is merely repeating (twice) the law of cosines to write:

$$b^2=a^2+c^2-2ac\cdot\cos\beta \qquad \text{XXXIX}$$

$$c^2=a^2+b^2-2ab\cdot\cos\gamma. \qquad \text{XL}$$

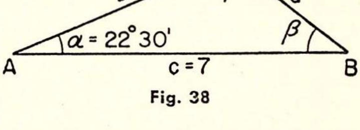

Fig. 38

Illustrative Example A

In a given triangle (Fig. 38), $\alpha=22°30'$, $b=5$, and $c=7$. Use the law of cosines to find a. Then use the law of cosines, once to find β and once to find γ. See whether $\alpha+\beta+\gamma$ is near 180°; if it is not, your work is not accurate.

a
$$a^2=5^2+7^2-(2\times5\times7\cdot\cos 22°30')=9.3284.$$
$$a=3.054.$$

b Since we now know a, b, and c, we can substitute in the formula,

$$b^2=a^2+c^2-2ac\cdot\cos\beta,$$

to obtain

$$25=9.3284+49-(2\times3.054\times7\cdot\cos\beta).$$

c $\therefore$ $\qquad \cos \beta = 0.77950$

and

$$\beta = 38°44'.$$

d We substitute in the formula,

$$c^2 = a^2 + b^2 - 2ab \cdot \cos \gamma,$$

to obtain

$$49 = 9.327 + 25 - 2 \times 3.054 \times 5 \cdot \cos \gamma.$$

e $\therefore$ $\qquad \cos \gamma = -0.48040.$

f Therefore γ is a second-quadrant angle, and $\gamma = 118°43'$.

g The sum of the computed values of α, β, and γ is $179°57'$.

If one wants more accurate results, one should use more extensive tables than those at the end of this volume.

Illustrative Example B

You may happen to have some reason for knowing the size of the greatest angle of the triangle having sides of lengths 5, 12, and 13. In any case, use the law of cosines to find the greatest angle, and check your result.

Letting θ denote the greatest angle, and using the fact that the greatest angle is opposite the longest side, we obtain

$$13^2 = 5^2 + 12^2 - 2 \times 5 \times 12 \cdot \cos \theta.$$

Hence, $\qquad \cos \theta = 0,$

so

$$\theta = 90°,$$

that is, θ is a right angle. That the side opposite θ is the hypotenuse of a right triangle is verified by the fact that

$$13^2 = 5^2 + 12^2.$$

TEST YOUR ABILITY TO USE THE COSINE LAW

72 With ruler and compasses, draw on graph paper a triangle (oriented as that in Fig. 31) whose sides have lengths $a=5$, $b=6$, $c=7$. Using the rulings on your graph paper, estimate the value of $\tan \alpha$ and then use the tables to obtain an estimate for α. Compute α by use of the law of cosines. Make your results check.

As one learns in physics and mechanics, it is important to be able to compute the length and direction of the diagonal of a parallelogram. As in Fig. 39, the lines, OA and OB, may represent forces, in which case the diagonal, OC, of the parallelogram would represent the resultant. Suppose $a=35$, angle $XOA = 18°$, $b=25$, and angle $XOB = 67°$. Find the length, c, of OC and the angle, XOC, in the following two ways, and make your answers check:

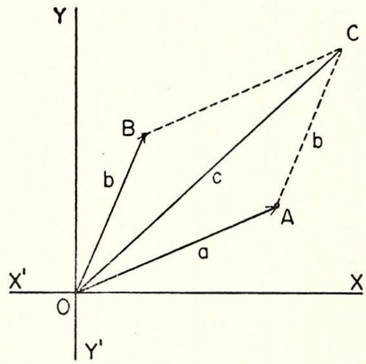

Fig. 39

73 Compute $\angle OAC$ and use the law of cosines to obtain c. Then use the law of sines to obtain $\angle AOC$ and add $\angle XOA$ to obtain $\angle XOC$.

74 Using $a=35$ and $\angle XOA = 18°$, compute the coördinates, x_A, and y_A, of A.

Similarly, compute the coördinates, x_B and y_B, of B. Show that the x- and y-coördinates of C are, respectively, x_A+x_B and y_A+y_B. Using the coördinates of C, compute c from the Pythagorean theorem and $\angle XOC$ from the value of its tangent.

The law of tangents

When two sides (say, b and c) of a triangle and the included angle, α, of a triangle are known, the law of cosines furnishes a method for computing the third side. The unknown angles may then be computed by the law of sines. The law of tangents enables us to compute the angles without first computing the unknown side. The *law of tangents* states that the sides and angles of an oblique or right triangle always satisfy the equation,

$$\frac{\tan \frac{1}{2}(\beta-\gamma)}{\tan \frac{1}{2}(\beta+\gamma)}=\frac{b-c}{b+c}. \qquad \text{XLI}$$

We shall not prove this; the simplest proof uses the law of sines and some identities from the next chapter.

When b, c, and α are known, the value of $\frac{1}{2}(\beta+\gamma)$ is readily computed since

$$\frac{1}{2}(\beta+\gamma)=\frac{1}{2}(180°-\alpha).$$

Hence, we can use XLI to compute $\tan \frac{1}{2}(\beta+\gamma)$ and hence, $\frac{1}{2}(\beta-\gamma)$. Adding $\frac{1}{2}(\beta+\gamma)$ and $\frac{1}{2}(\beta-\gamma)$, we obtain β; and subtracting gives γ.

TEST YOUR ABILITY TO USE THE TANGENT LAW

75 By use of the law of tangents, find all of the angles of the triangle for which $\alpha=22°30'$, $b=5$, and $c=7$.

The inscribed circle

We conclude our discussion of oblique triangles by giving a few relations involving the radius, r, of the inscribed circle.

Fig. 40 shows the circle inscribed in the triangle, ABC. The center of the circle is at O, the intersection of the bisectors of the interior angles of the triangles.

As indicated in the figure, let x, y, and z denote, respectively, the distances from A, B, and C to the points of tangency of the tangents from A, B, and C.

From the right triangle, AFO, with right angle at F and $\angle\frac{1}{2}\alpha$ at A, we see that

$$\tan \frac{1}{2}\alpha=\frac{r}{x}. \qquad \text{XLII}$$

Since $2x+2y+2z$ is the perimeter, $a+b+c$, of the triangle, we see that

$$x+y+z=\frac{1}{2}(a+b+c)=s, \qquad \text{XLIII}$$

where (as is customary in plane geometry and trigonometry) s is used to denote half the perimeter of the triangle. Since $y+z$ is the length, a, of the side opposite α, we can use XLIII to obtain

$$x=s-(y+z)=s-a \qquad \text{XLIV}$$

and then use XLII to obtain

$$\tan\frac{1}{2}\alpha=\frac{r}{s-a}. \qquad \text{XLV}$$

The corresponding formulas involving the other angles are

$$\tan\frac{1}{2}\beta=\frac{r}{s-b}, \qquad \text{XLVI}$$

$$\tan\frac{1}{2}\gamma=\frac{r}{s-c}. \qquad \text{XLVII}$$

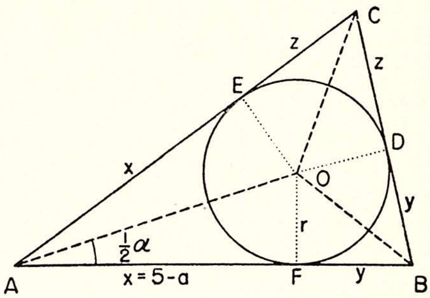

Fig. 40

When the sides and one angle of a triangle are known, these formulas enable us to compute the radius, r, of the inscribed circle.

It is shown in plane geometry that

$$r=\sqrt{\frac{(s-a)\ (s-b)\ (s-c)}{s}}. \qquad \text{XLVIII}$$

One way to find the angles of a triangle when a, b, and c are known is to compute (in order) the numbers, s, $s-a$, $s-b$, $s-c$, and r; and then use XLV and XLVI or XLVII.

TEST YOUR KNOWLEDGE OF INSCRIBED CIRCLES

76 Test this method by applying it to the triangle for which $a=5$, $b=6$, $c=7$. The results should check with those obtained by use of the law of cosines. When r and s have been computed, the area of the triangle is quickly obtained by use of a formula below.

There are many ways of finding the area, T, of the triangle, ABC. One way is to use the fact that the area, T, is the sum of the areas of the triangles, BOC, COA, and AOB, so that

$$T=\frac{1}{2}ra+\frac{1}{2}rb+\frac{1}{2}rc=r\frac{1}{2}(a+b+c) \qquad \text{XLIX}$$

and hence

$$T=rs. \qquad \text{L}$$

TRIGONOMETRIC IDENTITIES

After a section on right triangles, in which all angles considered lie between 0° and 90°, and a section on oblique triangles, in which all angles considered lie between 0° and 180°, we return to the general case, in which the measure of an angle may be either positive or negative (See page 385) and the terminal side of the angle may be in any quadrant (See page 395). All angles will be thought of as resting in the standard position on an xy-coördinate system.

The familiar construction is shown in Fig. 41 for the case in which θ is about 135°, or $\frac{3\pi}{4}$ radians. In substantially all applications of this section, θ stands for a number of radians. In such cases, one uses degree measure only when it is necessary to convert θ to degrees so that one may use tables featuring degree measure. When one has tables featuring radian measure, one uses radian measure exclusively. Accordingly, you will be preparing yourself for applications if you acquire the habit of thinking in terms of radian measure, using degree measure only when using the tables. All of the results of this chapter are obtained by starting with the definitions of the trigonometric functions, as given on page 395.

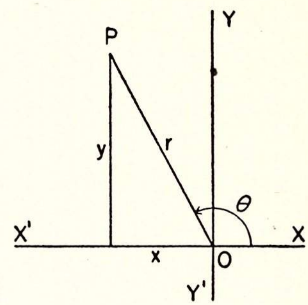

Fig. 41

It is possible to spend one's lifetime developing relations between the trigonometric functions. In this section, we confine our attention to the most important relations.

These relations are the most important because they have the most frequent applications in the pure and applied mathematics which one studies after completing a course in trigonometry. Fortunately, these relations are interesting. This means that we need not be bored by the studying we must do to prepare ourselves for the applications of trigonometry.

The fundamental identities

In the first place, as we have remarked before, the definitions show that three of the functions are reciprocals of the other three. We must know these formulas. Indeed, it is the usual practice to remember Formulas XXIII to XXV so well that they serve as an aid in recalling Formulas XX to XXII. If we divide Formula XVII by Formula XVIII (do this on paper), we obtain

$$\frac{\sin\theta}{\cos\theta}=\frac{y}{x}=\tan\theta,$$

which gives the formula,

$$\tan\theta=\frac{\sin\theta}{\cos\theta}. \qquad \text{LI}$$

This should be memorized so well that, whenever you see $\tan\theta$, you know it is $\frac{\sin\theta}{\cos\theta}$; and whenever you see $\frac{\sin\theta}{\cos\theta}$ you know it is $\tan\theta$. Whenever you see $\cot\theta$, you should know that this is $\frac{1}{\tan\theta}$ and hence is $\frac{\cos\theta}{\sin\theta}$.

The next three fundamental formulas are:

$$\sin^2\theta+\cos^2\theta=1, \qquad \text{LII}$$
$$1+\tan^2\theta=\sec^2\theta, \qquad \text{LIII}$$
$$1+\cot^2\theta=\csc^2\theta. \qquad \text{LIV}$$

Here, $\sin^2\theta$ means $(\sin\theta)^2$, the square of the number, $\sin\theta$. Everyone agrees to write $\sin^2\theta$ instead of $(\sin\theta)^2$ because it can be done more quickly. It is further agreed that, if n is an exponent different from -1, then $(\sin\theta)^n$ may be written in the form $\sin^n\theta$.

Caution: You must not write $(\sin\theta)^n$ in the form $\sin\theta^n$ because this would deceive everyone who reads your writing. The value of $\sin\theta^n$ is obtained by computing the nth power of θ (measured in radians) and taking the sine of the result. This is an operation very different from taking the sine of θ and computing the nth power of the result.

Incidentally, both $\sin^2\theta$ and $\sin\theta^2$ appear in problems in physics and engineering. The same remarks apply to powers of other trigonometric functions. The three formulas come from the fact that the coördinates, x and y, of a point on the terminal side of an angle, θ, are (by the Pythagorean theorem) such that

$$x^2+y^2=r^2, \qquad \text{LV}$$

where r is the distance from the origin to the point, P. If we divide by r^2, we get

$$\left(\frac{x}{r}\right)^2+\left(\frac{y}{r}\right)^2=1, \qquad \text{LVI}$$

and writing $\cos\theta$ and $\sin\theta$ for the quotients gives LII.

Similarly, dividing by x^2 gives LIII and dividing by y^2 gives LIV.

You should know these formulas so well that, whenever you see $\sin^2\theta+\cos^2\theta$, you think immediately of 1; and, whenever you see $1-\cos^2\theta$, you should know that it is $\sin^2\theta$.

USE OF THE FUNDAMENTAL IDENTITIES

The following little story is designed to illustrate the importance of knowing and being able to use the fundamental identities:

Suppose three persons, U (perhaps you), V, and W, are commissioned by an employer to solve and submit answers to a problem. Suppose the answers obtained, perhaps by different methods, are the following:

U's answer: $(\tan\theta+\cot\theta)^2$

V's answer: $\sec^2\theta\cdot\csc^2\theta$

W's answer: $\sec^2\theta+\csc^2\theta$.

Should it be concluded that two, or perhaps all three, of the solvers are untrustworthy? Even though the answers *look* different, it may be that (when θ is an angle for which the functions all exist) the numbers representing the

values of the expressions are not different. Taking *U*'s answer, and using fundamental identities, we find that

$$\begin{aligned}(\tan\theta+\cot\theta)^2&=\left(\frac{\sin\theta}{\cos\theta}+\frac{\cos\theta}{\sin\theta}\right)^2=\left(\frac{\sin^2\theta+\cos^2\theta}{\cos\theta\sin\theta}\right)^2\\&=\left(\frac{1}{\cos\theta\sin\theta}\right)^2=(\sec\theta\csc\theta)^2\\&=\sec^2\theta\csc^2\theta.\end{aligned}$$

Hence, the value of *U*'s answer is the same as that of *V*'s; the two agree.

Shall we conclude that *W* is in error; that he added when he should have multiplied? Taking *W*'s answer, we find that

$$\begin{aligned}\sec^2\theta+\csc^2\theta&=\frac{1}{\cos^2\theta}+\frac{1}{\sin^2\theta}=\frac{\sin^2\theta+\cos^2\theta}{\cos^2\theta\sin^2\theta}\\&=\frac{1}{\cos^2\theta\sin^2\theta}=\sec^2\theta\csc^2\theta\end{aligned}$$

and the value of *W*'s answer is the same as the value of *V*'s.

Thus, the values of the three answers are the same; the only difference is in appearance. If any one of the answers is correct, then all three are correct. There are still other ways in which one may express the answer; an individual, *X*, who knows the formulas presented later in the section could provide:

X's answer: $4\csc^2 2\theta$.

One who wishes an exercise in numerical computation can find the values of the four answers when θ has a given value, say $\theta=27°$. The four values are equal. If one were to use the four answers for numerical computation, one would say that the answer requiring the least computation is the best one.

When determining whether two expressions involving trigonometric functions are equal, it is often (but not always) best to commence by writing one or both of the expressions in terms of sines and cosines alone.

TEST YOUR KNOWLEDGE OF FUNDAMENTAL TRIGONOMETRIC IDENTITIES

By use of the fundamental identities, prove that

77 $\sin\theta\cot\theta=\cos\theta$

78 $\cos\theta\tan\theta=\sin\theta$

79 $(\sec\theta+\tan\theta)(\sec\theta-\tan\theta)=1$

80 $(\sin\theta+\cos\theta)^2=1+2\sin\theta\cos\theta$

81 $\sec\theta-\sin\theta\tan\theta=\cos\theta$

82 $\dfrac{1}{\tan\theta+\cot\theta}=\sin\theta\cos\theta$

83 $\dfrac{\sin\theta+\tan\theta}{1+\cos\theta}=\tan\theta$

84 Determine whether $\sec\theta-\tan\theta$ is equal to $\cos\theta$ for all values of θ. If you do not succeed in showing that they are always equal, perhaps you can show that they are not always equal by finding a value of θ for which they have different values.

The addition formulas

Let α and β be two angles. Then addition and subtraction give the angles, $\alpha+\beta$ and $\alpha-\beta$. For many reasons which cannot be appreciated at this time, one needs formulas relating the trigonometric functions of $\alpha+\beta$ and $\alpha-\beta$ to the trigonometric functions of α and β. Students learning trigonometry sometimes say that $\sin(\alpha+\beta)$ is always equal to $\sin\alpha+\sin\beta$; that the assertion is false is readily seen, by setting,

for example, $\alpha=\beta=90°$. The correct formulas, called the *addition formulas*, are

$$\sin(\alpha+\beta)=\sin\alpha\cos\beta+\cos\alpha\sin\beta \qquad \text{LVII}$$
$$\sin(\alpha-\beta)=\sin\alpha\cos\beta-\cos\alpha\sin\beta \qquad \text{LVIII}$$
$$\cos(\alpha+\beta)=\cos\alpha\cos\beta-\sin\alpha\sin\beta \qquad \text{LIX}$$
$$\cos(\alpha-\beta)=\cos\alpha\cos\beta+\sin\alpha\sin\beta \qquad \text{LX}$$
$$\tan(\alpha+\beta)=\frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta} \qquad \text{LXI}$$
$$\tan(\alpha-\beta)=\frac{\tan\alpha-\tan\beta}{1+\tan\alpha\tan\beta}. \qquad \text{LXII}$$

The first four formulas hold for all values of α and β. The last two hold for all values of α and β for which the trigonometric functions are defined.

If the formulas involving $\alpha+\beta$ are known, those involving $\alpha-\beta$ can be obtained from them by replacing β by $-\beta$ and using the fact that $\sin(-\beta)=-\sin\beta$, $\cos(-\beta)=\cos\beta$, and $\tan(-\beta)=-\tan\beta$. One way to become familiar with the six formulas is to write those involving $\alpha+\beta$ and then use them to obtain the other three. *One does not have satisfactory familiarity with these formulas until he can write out all six from memory.*

To work out the formula for $\sin(\alpha+\beta)$, we consider first the case in which α, β, and $\alpha+\beta$ are all acute angles.

The angle, $\alpha+\beta$, with terminal side, OP, is obtained by rotating a line from the initial position, OX, until it generates the angles, α and β, in succession.

Since $\alpha+\beta$ is a first-quadrant angle,

$$\sin(\alpha+\beta)=\frac{AP}{OP}.$$

In order to be able to express this quotient in terms of quotients of sides of right triangles of which α and β are angles, we draw PQ perpendicular to OQ so that $\triangle QOP$ is a right triangle including β, and then draw QC perpendicular to OX so that $\triangle COQ$ is a right triangle including α.

Drawing QB perpendicular to AP, we see that

$$BPQ=\alpha.$$

This gives Fig. 42. (You should draw the figure for yourself and be sure that you understand the construction.)

Since $AP=AB+BP=CQ+BP$, we obtain

$$\sin(\alpha+\beta)=\frac{CQ+BP}{OP}=\frac{CQ}{OP}+\frac{BP}{OP}.$$

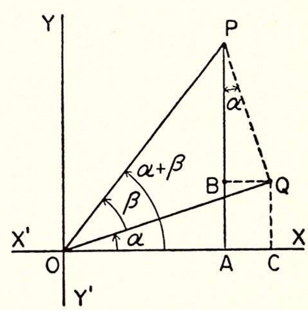

Fig. 42

Our next step is to work with the last quotients separately. We notice that the quotients do not represent trigonometric functions, but we can see that OQ forms a link between CQ and OP; since

$$\frac{CQ}{OQ}=\sin\alpha,$$
$$\frac{OQ}{OP}=\cos\beta,$$

we obtain

$$\frac{CQ}{OP}=\frac{CQ}{OQ}\cdot\frac{OQ}{OP}=\sin\alpha\cos\beta.$$

Similarly, QP forms a link between BP and OP:

$$\frac{BP}{OP}=\frac{BP}{QP}\cdot\frac{QP}{OP}=\cos\alpha\sin\beta.$$

$$\sin(\alpha+\beta)=\sin\alpha\cos\beta+\cos\alpha\sin\beta. \qquad \textbf{LXIII}$$

In case not all three of the angles, α, β, and $\alpha+\beta$, are positive acute angles, Fig. 42 would look different; but, in each case, the proof differs little from the proof for the case considered above. In more advanced mathematics (the theory of functions of a complex variable), the remaining cases are handled very easily: there is a theorem which implies that, if one of the addition formulas holds when α and β are small acute angles, then the formula holds for all angles, α and β, for which the functions involved are defined.

When α, β, and $\alpha+\beta$ are acute angles, the formula for $\cos(\alpha+\beta)$ may be obtained by use of Fig. 42; one begins with the formula,

$$\cos(\alpha+\beta)=\frac{OA}{OP}=\frac{OC-AC}{OP}=\frac{OC}{OP}-\frac{BQ}{OP}$$

and uses the links, OQ and QP, as before.

To obtain the formula for $\tan(\alpha+\beta)$, we commence with the formula,

$$\tan(\alpha+\beta)=\frac{\sin(\alpha+\beta)}{\cos(\alpha+\beta)}=\frac{\sin\alpha\cos\beta+\cos\alpha\sin\beta}{\cos\alpha\cos\beta-\sin\alpha\sin\beta}.$$

If we divide each term of the numerator and denominator of the last quotient by $\cos\alpha\cos\beta$, we obtain LXI.

Illustrative Example A

Let α and β be the second-quadrant angles for which $\sin\alpha=\frac{3}{5}$ and $\sin\beta=\frac{5}{13}$. Find the sine, cosine, and tangent of $\alpha+\beta$ and $\alpha-\beta$.

By drawing figures, we find that

$$\cos\alpha=-\frac{4}{5},\ \tan\alpha=-\frac{3}{4},\ \cos\beta=-\frac{12}{13},\ \tan\beta=-\frac{5}{12}.$$

Hence, use of the addition formulas gives:

$$\sin(\alpha-\beta)=\left(\frac{3}{5}\right)\left(-\frac{12}{13}\right)-\left(-\frac{4}{5}\right)\left(\frac{5}{13}\right)=-\frac{16}{65},$$

$$\cos(\alpha-\beta)=\left(-\frac{4}{5}\right)\left(-\frac{12}{13}\right)+\left(\frac{3}{5}\right)\left(\frac{5}{13}\right)=\frac{63}{65},$$

$$\tan(\alpha-\beta)=\frac{\left(-\frac{3}{4}\right)-\left(-\frac{5}{12}\right)}{1+\left(-\frac{3}{4}\right)\left(-\frac{5}{12}\right)}=-\frac{16}{63}.$$

Illustrative Example B

Find sin 15° by expanding $\sin(45°-30°)$.
Using an addition formula gives:

$$\sin(45°-30°)=\sin 45°\cos 30°-\cos 45°\sin 30°$$
$$=\frac{\sqrt{2}}{2}\cdot\frac{\sqrt{3}}{2}-\frac{\sqrt{2}}{2}\cdot\frac{1}{2}=\frac{\sqrt{6}-\sqrt{2}}{4}=0.25882$$

TEST YOUR KNOWLEDGE OF ADDITION FORMULAS

Find

85 $\sin 75°$ **86** $\cos 75°$ **87** $\tan 75°$

by expanding the functions of $(45°+30°)$.

Check your answers by showing that

88 $\sin^2 75°+\cos^2 75°=1$ **89** $\cos 75°=\sin 15°$

Expand each of the following functions by use of an addition formula and simplify the result by use of the known values (page 441) of the functions of multiples of 90°:

90 $\sin(90°-\theta)$ **92** $\sin(90°+\theta)$ **94** $\sin(180°-\theta)$

91 $\cos(90°-\theta)$ **93** $\cos(90°+\theta)$ **95** $\cos(180°-\theta)$

Make your results check with those on pages 442 and 443.

FUNCTIONS OF MULTIPLE ANGLES

In terms of a given angle, θ, the angle, 2θ, is commonly called a *double-angle* and the angle, $\frac{1}{2}\theta$, a *half-angle.*

The *double-angle* formulas are those which express functions of 2θ in terms of functions of θ; we obtain them by setting $\alpha=\beta=\theta$ in Formulas LVII, LIX, and LXI:

$$\sin 2\theta=2\sin\theta\cos\theta \qquad \textbf{LXIV}$$
$$\cos 2\theta=\cos^2\theta-\sin^2\theta \qquad \textbf{LXV}$$
$$\tan 2\theta=\frac{2\tan\theta}{1-\tan^2\theta}. \qquad \textbf{LXVI}$$

Illustrative Example

Express $\sin 3\theta$ in terms of $\sin\theta$.

$$\sin 3\theta=\sin(2\theta+\theta)=\sin 2\theta\cos\theta+\cos 2\theta\sin\theta$$
$$=(2\sin\theta\cos\theta)\cos\theta+(\cos^2\theta-\sin^2\theta)\sin\theta$$
$$=2\sin\theta(1-\sin^2\theta)+(1-2\sin^2\theta)\sin\theta$$
$$=3\sin\theta-4\sin^3\theta.$$

Formulas relating products and sums

Adding Formulas LVII and LVIII for $\sin(\alpha+\beta)$ and $\sin(\alpha-\beta)$ gives:

$$\sin(\alpha+\beta)+\sin(\alpha-\beta)=2\sin\alpha\cos\beta \qquad \textbf{LXVII}$$

and subtracting LVIII from LVII gives

$$\sin(\alpha+\beta)-\sin(\alpha-\beta)=2\cos\alpha\sin\beta. \qquad \textbf{LXVIII}$$

Adding Formulas LIX and LX for $\cos(\alpha+\beta)$ and $\cos(\alpha-\beta)$ gives

$$\cos(\alpha+\beta)+\cos(\alpha-\beta)=2\cos\alpha\cos\beta, \qquad \textbf{LXIX}$$

and subtracting LX from LIX gives

$$\cos(\alpha+\beta)-\cos(\alpha-\beta)=-2\sin\alpha\sin\beta. \qquad \textbf{LXX}$$

A set of three especially useful formulas is obtained from LXVII, LXIX, and LXX by dividing them by 2 and transposing. The formulas are

$$\sin \alpha \cos \beta = \frac{1}{2}\left[\sin(\alpha-\beta)+\sin(\alpha+\beta)\right] \qquad \text{LXXI}$$

$$\cos \alpha \cos \beta = \frac{1}{2}\left[\cos(\alpha-\beta)+\cos(\alpha+\beta)\right] \qquad \text{LXXII}$$

$$\sin \alpha \sin \beta = \frac{1}{2}\left[\cos(\alpha-\beta)-\cos(\alpha+\beta)\right]. \qquad \text{LXXIII}$$

It is nearly always true in applications of trigonometry that products of sines and cosines are much more difficult to handle than sums and differences of sines and cosines. For example, in the calculus, the problem of integrating $\sin mx \cos nx$ is made very simple by use of the fact that

$$\sin mx \cos nx = \frac{1}{2}\left[\sin(m-n)x+\sin(m+n)x\right].$$

Despite the usefulness of LXXI, LXXII, and LXXIII, it is not necessary to remember them; when one wants to express a product as a sum, one can either work out the appropriate formula or find it in a book.

It is worth while to remember the two formulas to which LXXII and LXXIII reduce when $\alpha=\beta$. Setting $\alpha=\beta=\theta$ in LXXII and LXXIII gives

$$\cos^2\theta = \frac{1+\cos 2\theta}{2}, \quad \sin^2\theta = \frac{1-\cos 2\theta}{2}. \qquad \text{LXXIV}$$

When one knows these formulas, he can replace θ by $\frac{1}{2}\varphi$ and take square roots to obtain

$$\cos\frac{\varphi}{2} = \pm\sqrt{\frac{1+\cos\varphi}{2}}, \qquad \text{LXXV}$$

$$\sin\frac{\varphi}{2} = \pm\sqrt{\frac{1-\cos\varphi}{2}}. \qquad \text{LXXVI}$$

These are *half-angle formulas*. Division gives also

$$\tan\frac{\varphi}{2} = \pm\sqrt{\frac{1-\cos\varphi}{1+\cos\varphi}}. \qquad \text{LXXVII}$$

In each of LXVI and LXVII, one can determine whether the + or − sign should be used when one knows the quadrant in which $\frac{1}{2}\varphi$ lies.

There is another way in which one can obtain interesting (and sometimes useful) formulas from LXVII, LXVIII, LXIX, and LXX.

If x and y are measures of angles, we can determine α and β so that

$$\alpha+\beta=x \text{ and } \alpha-\beta=y.$$

Adding gives $\qquad \alpha=\frac{1}{2}(x+y)$

and subtracting gives $\qquad \beta=\frac{1}{2}(x-y).$

Substituting these values of α and β in the formulas, LXVII, LXVIII, LXIX, and LXX gives

$$\sin x+\sin y=2\sin\frac{1}{2}(x+y)\cos\frac{1}{2}(x-y) \qquad \text{LXXVIII}$$

$$\sin x-\sin y=2\cos\frac{1}{2}(x+y)\sin\frac{1}{2}(x-y) \qquad \text{LXXIX}$$

$$\cos x+\cos y=2\cos\frac{1}{2}(x+y)\cos\frac{1}{2}(x-y) \qquad \text{LXXX}$$

$$\cos x-\cos y=-2\sin\frac{1}{2}(x+y)\sin\frac{1}{2}(x-y). \qquad \text{LXXXI}$$

These formulas need not be remembered; if you want the formulas for sums and differences of sines (or of cosines), look them up.

Illustrative Example A

If θ is a third-quadrant angle and $\cos\theta=0.6$, find $\sin\frac{1}{2}\theta$, $\cos\frac{1}{2}\theta$, and $\tan\frac{1}{2}\theta$.

Since θ is in the third quadrant, $180°<\theta<270°$; so $90°<\frac{1}{2}\theta<135°$ and $\frac{1}{2}\theta$ is in the second quadrant. Hence,

$$\sin\frac{\theta}{2}=\sqrt{\frac{1-\cos\theta}{2}}=\sqrt{0.2};$$

$$\cos\frac{\theta}{2}=-\sqrt{\frac{1+\cos\theta}{2}}=-\sqrt{0.8};$$

$$\tan\theta=-\sqrt{\frac{1-\cos\theta}{1+\cos\theta}}=-\sqrt{\frac{0.4}{1.6}}=-\frac{1}{2}.$$

Illustrative Example B

Using the fact that 22°30′ is half of 45°, find the sine, cosine, and tangent of 22°30′.

All functions of 22°30′ are positive, and

$$\sin 22°30'=\sqrt{\frac{1-\cos 45°}{2}}=\frac{1}{2}\sqrt{2-\sqrt{2}},$$

$$\cos 22°30'=\sqrt{\frac{1+\cos 45°}{2}}=\frac{1}{2}\sqrt{2+\sqrt{2}},$$

$$\tan 22°30'=\sqrt{\frac{1-\cos 45°}{1+\cos 45°}}=\sqrt{\frac{2-\sqrt{2}}{2+\sqrt{2}}}=\sqrt{2}-1.$$

TEST YOUR KNOWLEDGE OF RELATING PRODUCTS AND SUMS

For each of the following equations, show how one may begin with the left member and prove that it is equal to the right member:

96 $\sin 40°\cos 20°=\frac{1}{2}(\sin 20°+\sin 60°)$

97 $\cos 42°\cos 24°=\frac{1}{2}(\cos 18°+\cos 66°)$

98 $\sin 33°\sin 11°=\frac{1}{2}(\cos 22°-\cos 44°)$

99 $\sin 16°+\sin 14°=2 \sin 15° \cos 1°$
100 $\sin 16°-\sin 14°=2 \cos 15° \sin 1°$
101 $\cos 60°+\cos 30°=2 \cos 45° \cos 15°$
102 $\cos 20°-\cos 10°=-2 \sin 15° \sin 5°$
103 Use the formula, $\sin^2\theta+\cos^2\theta=1$, and the formula for $\cos 2\theta$ to obtain the formulas in LXXIV. Use the tables to find the value of each member when $\theta=11°$, and thus determine whether it is easier to find the values of the left or the right members.

INVERSE TRIGONOMETRIC FUNCTIONS

We know that, if $x>1$ or <-1, then there is no angle, θ, such that $\sin\theta=x$. If $-1 \leqq x \leqq 1$, then there are many angles, θ, such that $\sin\theta=x$; but there is *exactly one* angle θ such that $-\frac{1}{2}\pi \leqq \theta \leqq \frac{1}{2}\pi$ and $\sin \theta=x$. This motivates introduction of the symbol, $\sin^{-1}x$, sometimes read "inverse sine of x" and sometimes read "angle whose sine is x". If $x>1$ or $x<-1$, the symbol is meaningless; but if $-1 \leqq x \leqq 1$, the symbol stands for the one and only angle, θ, such that

$$-\frac{1}{2}\pi \leqq \theta \leqq \frac{1}{2}\pi$$

and

$$\sin \theta=x.$$

For example, $\sin^{-1}\frac{1}{2}=\frac{1}{6}\pi(=30°)$ and $\sin^{-1}\left(-\frac{1}{2}\right)=-\frac{1}{6}\pi(=-30°)$.

As x begins at -1 and increases to 1, the angle, $\sin^{-1}x$, *increases continuously* from $-\frac{1}{2}\pi$ to $\frac{1}{2}\pi$.

Similarly, the symbol, $\cos^{-1}x$ is meaningless when $x>1$ and when $x<-1$; but if $-1 \leqq x \leqq 1$, the symbol stands for the one and only angle, θ, such that $0 \leqq \theta \leqq \pi$ and $\cos \theta=x$.

For example,

$$\cos^{-1}\frac{1}{2}=\frac{1}{3}\pi(=60°)$$

and

$$\cos^{-1}\left(-\frac{1}{2}\right)=\frac{2}{3}\pi(=120°).$$

As x begins at -1 and increases to $+1$, the angle, $\cos^{-1}x$, *decreases continuously* from π to 0.

For each number, x, the symbol, $\tan^{-1} x$, denotes the one and only angle, θ, such that

$$-\frac{1}{2}\pi<\theta<\frac{1}{2}\pi$$

and

$$\tan \theta=x.$$

For example,

$$\tan^{-1} 1 = \frac{1}{4}\pi \left(=45^\circ\right)$$

and

$$\tan^{-1}(-1) = -\frac{1}{4}\pi\left(=-45^\circ\right).$$

As x begins from $-1{,}000{,}000$ and increases to $+1{,}000{,}000$, the angle, $\tan^{-1}x$, *increases continuously* from a value just a little greater than $-\frac{1}{2}\pi$ to a value just a little less than $\frac{1}{2}\pi$.

The definitions of $\cot^{-1}x$, $\sec^{-1}x$, and $\csc^{-1}x$ are similar; the definitions imply that, for appropriate values of x,

$$\cot^{-1} x = \tan^{-1}\left(\frac{1}{x}\right),$$

$$\sec^{-1} x = \cos^{-1}\left(\frac{1}{x}\right),$$

$$\csc^{-1} x = \sin^{-1}\left(\frac{1}{x}\right).$$

When x is negative, the quadrant in which an inverse function of x lies depends upon the inverse function; but, when x is positive, the angles, $\sin^{-1}x$, $\cos^{-1}x$, $\tan^{-1}x$, etc., are all positive acute angles.

The definitions are (like all *good* definitions) designed to be useful. For each x in an appropriate interval, each of the functions, $\sin^{-1}x$, etc., has an unambiguous value, and the value changes continuously as x ranges continuously over the interval. The reader is not expected to appreciate the inverse trigonometric functions fully until he sees (in the calculus and its applications) how they arise naturally in the solution of practical problems.

TEST YOUR KNOWLEDGE OF INVERSE FUNCTIONS

Using the definitions of the inverse trigonometric functions, show that

104 $\sin^{-1} 1 - \sin^{-1} 0 = \frac{1}{2}\pi - 0 = \frac{1}{2}\pi.$

105 $\sin^{-1} 1 - \sin^{-1}(-1) = \frac{1}{2}\pi - \left(-\frac{1}{2}\pi\right) = \pi.$

106 $\cos^{-1} 1 - \cos^{-1}(-1) = 0 - (-\pi) = \pi.$

107 $\tan^{-1} 1 - \tan^{-1}(-1) = \frac{1}{4}\pi - \left(-\frac{1}{4}\pi\right) = \frac{1}{2}\pi.$

108 $\sin^{-1}\left(-\frac{1}{2}\right) - \cos^{-1}\left(-\frac{1}{2}\right) = -\frac{1}{6}\pi - \frac{2}{3}\pi = -\frac{5}{6}\pi.$

109 $\tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{3} = \frac{\pi}{4}$ [Hint: $\tan(\tan^{-1}x) = x$.]

A FINAL CHECK ON TRIGONOMETRIC KNOWLEDGE

110 One who wishes a more severe test of his powers may undertake to establish the famous formula (due to Machin):

$$4\tan^{-1}\frac{1}{5} - \tan^{-1}\frac{1}{239} = \frac{\pi}{4}.$$

THE SLIDE RULE IN TRIGONOMETRY

by Herbert Harvey

HOW the slide rule is used in the ordinary operations of arithmetic was explained at length in Issue Number Two.

This versatile assistant is by no means limited in its usefulness to elementary problems. In trigonometry, we regularly have to perform multiplications and divisions involving sines, cosines, and the other trigonometric functions. Slide rules may be equipped, and the majority of slide rules on the market are equipped, with scales which can be used as conveniently for these operations of trigonometry as the *A*-, *B*-, *C*-, and *D*-scales are for arithmetic.

THE TRIGONOMETRIC SCALES

The trigonometric scales are two in number, and are found on the *back* of the slide. (The slide rule in its ordinary position, with the *A*-, *B*-, *C*-, and *D*-scales in view, is shown in Issue Number Two, page 102.)

The slide rule with slide reversed is shown in Fig. 1. The trigonometric scales are marked *S* and *T*. The third scale in the center, marked *L*, is divided linearly, each scale division being $\frac{1}{500}$ of the scale length.

Take your rule with the trigonometric scales showing, and set 0 of the *L*-scale opposite 1 of the *D*-scale. This is the "zero" position. Now place the runner so that the hairline is on 2 of the *D*-scale, and read off the *L*-scale. The reading is 0.3010 (Fig. 1).

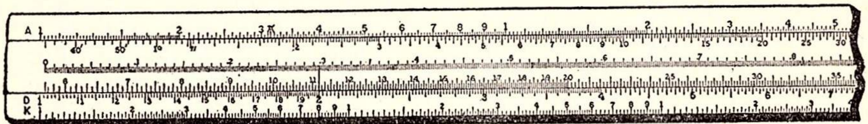

Fig. 1

This is the logarithm of 2. By setting the hairline at various numbers on the *D*-scale, we readily see that *L* always gives the (mantissa of) the logarithm of the number on the *D*-scale. This agrees with the fact, explained in Issue Number Two, that the *D*-scale is laid off logarithmically (and of course is why the two scales, *C* and *D*, add logarithms and, therefore, multiply numbers). Similarly, the *A*-scale is laid off logarithmically, but with *double* the reading of the *L*-scale being the logarithm of the number, so that it is the square root of the reading on the *A*-scale that has its logarithm opposite it on the *L*-scale (and, of course, the square root itself opposite it on the *D*-scale).

Construction of the scales

The trigonometric tangents of angles between 0° and 45° lie between 0 and 1.00. If we take the number, 1, on the right end of the *D*-scale to represent 1.00, then the 1 on the left end of the *D*-scale represents 0.1 (this is equivalent to imputing the characteristic, $\bar{1}$ or 9-10, to mantissas in the *L*-scale.) This is the basis of the *T*-scale.

Tan 45° = 1.00, and 45° on the *T*-scale is opposite 1 at the *right* end of the *D*-scale (Fig. 1). Tan 6° = 0.1051, and we find 6 on the *T*-scale just a little past the 1 on the *left* end of the *D*-scale—in fact, at 1051 (Fig. 1). For all angles between about 6° (more exactly, 5°43′) and 45°, the *angles* appear on the *T*-scale and the *tangents* on the *D*-scale.

Without altogether forgetting them, we shall put aside for the moment angles less than 6°, and confine our attention to the angles between 6° and 45° which the scales cover. What we have just shown is that, just as the *D*-scale was laid out proportionally to the logarithms of the numbers, so the *T*-scale is laid out proportionally to the *logarithms of the tangents* of the angles from 6° to 45°.

Without moving the slide from the position in Fig. 1, we may read from the rule directly the tangent of any angle between 6° and 45° by placing the hairline so as to read the *angle* on the *T*-scale, and then read off the *tangent* directly on the *D*-scale. The decimal point may be located by bearing in mind that the tangent of 45° = 1. Thus, from Fig. 2, we see that tan 11°19′ = 0.200.

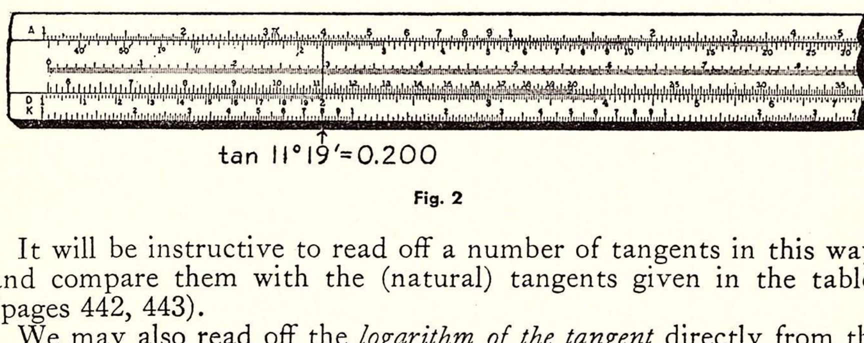

Fig. 2

It will be instructive to read off a number of tangents in this way, and compare them with the (natural) tangents given in the tables (pages 442, 443).

We may also read off the *logarithm of the tangent* directly from the scales. This is done by setting the hairline to read the angle on the *T*-scale, and reading the mantissa from the *L*-scale. Test this for various angles by comparing the slide rule readings with the log tangent table.

The *S*-scale is laid off opposite the roomier cycles of the *A*-scale. The numbers represented on the *A*-scale may be pointed off so that the 1 at the *right* end represents 1.00 and the 1 at the *left* end represents 0.01. By laying off the *S*-scale opposite this scale, it is therefore possible to represent the sines between 0.01 and 1.00, which are the sines of angles between 0°34′23″ and 90°. This is the range of angles covered on the *S*-scale. Again, for the moment, we lay aside angles

of less than 0°35′ as we shall find that provision for them is made in another way.

Thus, the *S*-scale is graduated proportionally to the *logarithm of the sine* of the designated angle, much as the *T*-scale is proportional to the log tangent of the designated angle.

From the *S*- and *A*-scales, we may read directly the natural sine of any angle between 35′ and 90°. Simply place the hairline to mark the angle on the *S*-scale, and read the sine on the *A*-scale. Thus, in the position shown in Fig. 1, we read, sin 2°18′ = 0.040. It is suggested that you make several readings on your slide rule of the sines of various angles in the same way, paying particular attention to the method of locating the decimal point.

Reading the scales

In reading angles, especially on the *S*-scale, a few hints on the way the scale is graduated will be helpful.

First let us look again at the *T*-scale. Here all the numbers represent degrees. At the left end of the scale, we find each integer degree printed in, with each degree graduated into twelfths. (The reason for using the twelfth is, of course, the non-decimal division of the angles into sixty minutes. Each twelfth represents 5′.) At the right end of the *T*-scale, every fifth degree is marked with an integer printed on the scale, and each degree is graduated into sixths, or one scale division for each 10′. This makes it quite easy to read angles off the *T*-scale.

The *S*-scale covers a wider range of angles and of sines than the *T*-scale. At the left end of the *S*-scale, we have an angle of less than 1°. Each scale division (on the rule illustrated in our figures) represents 2′ in this part of the *S*-scale. Notice that 1° on the *S*-scale appears a little to the right of this end, about opposite 174 on the *A*-scale (the sine of 1° is 0.0174).

Continuing to the right along the *S*-scale from 1°, we come to 2° considerably farther along, at 349 on the *A*-scale. (The sine of 2° is 0.0349.) Between 1° and 2° on this scale, we find the interval divided rather finely, each scale division proving, on examination, to represent 2′. (You will notice in this section of the scale two marks, one ′ and the other ″. We shall have use for these later.)

Proceeding farther to the right, we find that the subdivision of the *S*-scale continues in the same way to 3°. The finer graduations are dropped after 3°, the angles between 3° and 10° being subdivided only to each 5′. Each integer degree is printed on the scale up to 10°.

Between 10° and 70°, the scale becomes rapidly tighter, and the fine-graduation is progressively altered, until, at 70°, each scale division represents 1°. Between 70° and 90° (the latter opposite the 1 at the right end of the *S*-scale) the continual foreshortening of the log scale, coupled with the more rapid approach of the sine to its limit of 1.00, bunches the graduations together so that a setting in this part of the scale can be made only to the nearest degree or so. This makes no difference when it comes to reading the sine from the *A*-scale—the sines vary little here from degree to degree.

Notice that we cannot read log sine on the *L*-scale in the manner in which we read log tangent on it in connection with the T-scale.

When you are sure that you know how the S-scale, proportional to log sin x, and the T-scale, proportional to log tan x, where x is the scale reading, are constructed, and that you can read on them correctly to give the accuracy in angle, sine, tangent, and log tangent for which your rule is designed, reckoning with any slight differences which may exist between the design of the rule you are using and the rule shown in our diagrams, proceed with the illustrative examples. More about the slide rule in trigonometrical problems will appear later.

Illustrative Examples

A To multiply a number by a tangent.

Find 5 tan 31°. Set the *right* end of the T-scale on 5.0 of the D-scale. Set the runner so that the hairline reads 31° on the T-scale. Under the hairline, we may then read the product on the D-scale, 3.004.

In all multiplication problems the slide rule gives the significant figures of the answer; the decimal point has to be located by inspection. The previous article explains this.

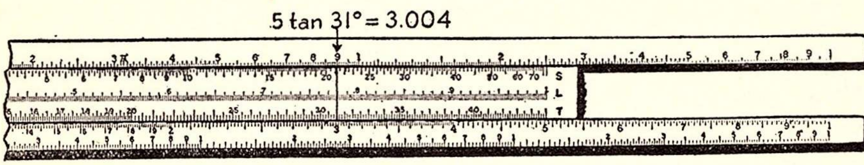

Fig. 3

B Find $3.72 \cdot \tan^2 31°$.

Set the right end of the T-scale under the reading, 372, on either half of the A-scale. (This is done by setting the right end of the S-scale there, since all scales end on the same line.) Now set the hairline to read 31° on the T-scale. Under the hairline, read the product on the A-scale, 1.343.

C Find $6.18 \cdot \sin 3°20'$.

Set the right end of the S-scale on 618 of the A-scale. Set the hairline to read 13°20′ on the S-scale, and find the reading, 1425, on the A-scale. The product is 1.425.

D To find 1.5 cot 17°.

Cot 17° is $\frac{1}{\tan 17°}$. Hence set the hairline to read 1.5 on the D-scale; next operate the slide to read 17° on the T-scale under the hairline; the answer will be found on the D-scale opposite the end of the T-scale (4.90).

E To find $12 \cdot \cos 12°$.

We know that $\cos 12 = \sin 78°$. We therefore multiply 12 by sin 78° (as in example C) and the product is $12 \cos 12° = 11.7$.

F To find $\sin 30° \cdot \sin 23°$.

With the slide in the zero-position (the position in Fig. 1) set the hairline to read 30° on the S-scale. Next operate the slide until the right end of the S-scale is on the hairline. With the slide in this position, move the runner until the hairline reads 23° on the S-scale. The hairline will now read the product on the A-scale (0.195).

The Measuring Rod

1 The sides of a triangular field are 3, 5, and 7 rods. Find (without the use of tables) the value of the largest angle.

2 When a coil of wire is rotated between the ends of a magnet in a loud-speaker and the poles of a generator, an alternating voltage is generated. This is represented by the formula, $e = E \sin \theta$, where e is the instantaneous voltage at a given moment, E is the maximum voltage, and θ is the phase angle through which the coil has been rotated. What is the instantaneous voltage, when $E = 6.4$ and θ is 143°?

3 The power (P) in watts for an alternating current in circuit is given by the formula, $P = EI \cos \theta$, where E and I are the effective values for the electromotive force and current respectively, and θ is the phase angle. Find θ in radians, if E is 120 volts, I is 2 amperes, and P is 189 watts

4 A sailor at sea is at a distance of 110 miles from a mountain (See Fig. 43) when the top of the mountain is just visible. How high is the mountain? (Take the radius of the earth as 4000 miles.)

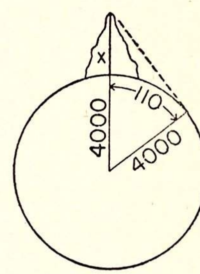

Fig. 43

5 An approximation often used to find the length of an uncrossed belt joining two pulleys is: the sum of twice the distance of centers and one-half the sum of the circumferences of the two pulleys. How great an error would result from this approximation if the pulleys are 28 and 44 inches in diameter and the distance of centers is 5 feet?

6 Draw a sketch and explain the method by which you would measure the area of an irregular ten-sided plot of ground.

7 An ammunition train is running due east on a straight railroad track. From a point, P, on the track, a straight road extends 36° east of north to a tower, Q, where there is a battery of guns whose range is 34 miles. If PQ is 40 miles, how many miles of track are in danger of attack?

8 A horizontal bridge 208.4 ft. long is built across a ravine. From a certain point, A, at the bottom of the ravine and directly under the bridge a pier is to be erected supporting the bridge. If the angles of elevation of the two ends of the bridge as seen from the point, A, are 57°23′ and 68°13′, find the height of the pier.

9 A funnel is to be made from a length of wood. After the wood has been turned on a lathe to the proper shape, the diameter of the mouthpiece is 2″ and the length is $3\frac{1}{4}''$. If the tube is to be $\frac{3}{8}''$ in diameter and the sides $\frac{1}{16}''$ thick, at what angle should the cutting tool be fed to bore out the mouthpiece? (See Fig. 44.)

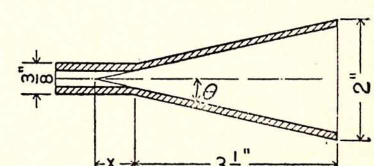

Fig. 44

10 A garage is 12′ high and fixed in its top is a flagpole 15′ high. On the opposite side of the street from the garage at a given point, the garage and the flagpole subtend equal angles. How wide is the street?

11 Two lights are 28.4 miles and 16.3 miles respectively from a ship. The first light bears 37°19′ east of north from the ship, and the second bears 59°49′ east of north from the ship. Find to the nearest tenth of a mile the distance between the lights.

12 What is the altitude of the sun expressed as an inverse function of the angle of elevation, when an object casts a shadow twice its own height?

13 Along the bank of a stream following a straight line, a distance of 200 feet is measured. The angles between this line and the lines of sight from its extremities to a tree on the opposite bank of the stream are 23°12′ and 79°42′. Find the width of the stream. (Fig. 45)

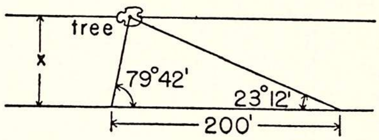

Fig. 45

14 A house is situated on a hillside which is inclined 12°39′ to the horizontal plane. A ladder 32.75′ long just reaches the bottom of a window 25′ from the ground. What is the distance of the foot of the ladder from the house, measured down the slope?

15 An enemy tank of a type known to be 12 feet long subtends an angle of 2 mils. What is the range?

16 From the top of a lighthouse 200′ above sea level, the angle of depression of a passing ship was 10°14′, and five minutes later the angle of depression was 11°10′. The angle between the two directions of the ship as seen from the foot of the lighthouse was 127°14′. Find the distance the ship traveled during those five minutes.

17 From the top of a lighthouse 85 ft. high standing on a rock, the angle of depression of a ship was 36°15′, and, from the bottom of the lighthouse, the angle of depression was 10°45′. What was the height of the rock? (Fig. 46)

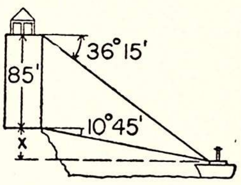

Fig. 46

18 On one bank of a stream, a wall 15′ high supports a flagstaff 21′ high. From a point on the other bank, directly opposite the flagstaff and on a level with the bottom of the wall, the flagstaff and wall subtend equal angles. Find the width of the stream.

19 A rectangular sheet of aluminum 3 feet by 8 feet is to be cut from one corner to an opposite side so that the angle cut from the right angle is 51.5°. What is the other point determining the cut?

20 A conical cistern has a vertex angle of 70°. At what depth will 18 cubic feet of water stand?

21 A wedge-shaped vat for chemicals is to have a depth of 4 feet. If the angle at the bottom must be 126°, and the vat is to contain 487 cu. ft. of chemicals, what must its length be?

22 A military observer notes two enemy batteries which subtend, at his observation post, an angle of 42°. The interval between the flash and the report of a gun is 3 seconds for one battery and 4 seconds for the other. If the velocity of sound is 1140 feet per second, how far apart are the batteries?

23 A surveyor desires to extend a straight line, AB, due east past an obstruction, H. He measures BC 792.6 feet S. 27°14′ E., and then runs CD N. 46°57′ E. Find the length of CD so that D will be due east of B.

24 A gunner at G can see a point, P, (called the aiming point) but cannot see the target, T. The distances, G and PT, and the angle, GPT, are measured and found to be 3,820 yards, 8,405 yards, and 133°52′, respectively. (Angle PGT is called the angle of deflection and the distance, GT, is called the range.) Find the angle of deflection correct to the nearest minute and the range correct to the nearest yard.

25 An airplane is traveling at a rate of 120 m.p.h. on a straight course parallel to the earth. At a certain time, the angle of elevation, θ, of the plane is 43° and 2 minutes later the angle of elevation, φ, is 20°. Show that the height, h, of the airplane above the ground is given by the formula, $h=\dfrac{d}{\cot\varphi-\cot\theta}$, where d is the distance traveled by the plane. Using this formula, find the height in feet of the plane.

26 In bracing a 40-foot telegraph pole, a wire is to be attached to the pole 35 feet up. If the wire is 60 feet long and the ground slopes down from the base of the pole at an angle of 16°, how far from the base of the pole should the anchor be? (Fig. 47)

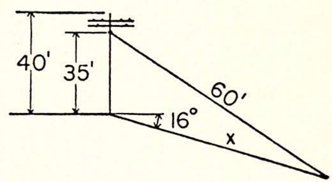

Fig. 47

27 A three-inch gun with a muzzle velocity of 1600 feet per second is aimed at an enemy gun battery 12 miles away. Find the angle of elevation (θ) of the gun. (Use the formula, $y=\dfrac{-32x^2}{2V_0^2\cos^2\theta}+x\tan\theta$, in which x is the range of the gun, V_0 the initial velocity, and y the height to which the projectile will rise.)

28 What is the area of a triangular field, two of whose sides measure 86 and 34 rods, if the angle between them is 49.3°?

29 The diameter of a flywheel is 7″ and it revolves 63 times in 1 minute. What is its rim speed?

30 Two ships start together from port. One steams 15° E. of S. at 18 knots and the other 65° E. of N. at 14 knots. Find the speed at which they are receding from each other.

31 If the legs of a pair of dividers are 7 inches long, at what angle must they be set so that the distance between the points is 9.23 inches?

32 The altimeter on his airplane having been broken, a pilot flying directly over town A takes a sight on town B which is known to be 19 miles from A. If the angle of depression is 26°, what is the altitude of the plane? (Fig. 48)

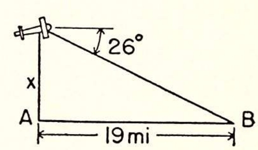

Fig. 48

33 A mountain is to be tunneled. Draw a sketch and explain the method by which you would find the length of the proposed tunnel if its ends were determined.

34 A navigator in the desert tank corps makes an error of 3 degrees in his course. At the end of 20 miles, how far is he off his course?

35 Find the angle at which a boat will cross a river if the river flows 3 miles per hour and the boat is rowed pointing perpendicularly across at 4 miles per hour?

Solutions to Problems and Exercises in Issue 6

SOLID GEOMETRY

FUNDAMENTAL ELEMENTS

Planes

1 Use property of a plane that a straight line lies in it with all of its points (regardless of how the line is placed).

2 Because the 3 end-points of the legs of the tripod will determine a plane.

3 They have four legs, the ends of which determine more than one plane.

5 No.

Intersecting Planes

6 A straight line.

7 When he pulls the string, he forms a plane. The intersection of that plane with the board is a straight line.

8 An infinity.

9 No. They may be cylinders or cones.

10 Yes. The 3 vertices determine a plane.

11 Board and saw are intersecting planes.

Perpendiculars

12 Yes.

13 The figure is an isosceles triangle. The vertical string is perpendicular to the base line of the level.

14 No; for the same reason that two perpendiculars can not be drawn from the same point to a line.

15 No. **16** They are parallel.

Oblique Lines

17 The lines are inclined at the same angle to the plane.

18 Yes. **19** Yes.

20 Along a common perpendicular line.

POINTS IN SPACE

Locating Points

46 $\sqrt{(0-4)^2+(3-0)^2+(5-5)^2}=5$

47 $\sqrt{27}=3\sqrt{3}=5.196$

48 $(x-1)^2+50=100;\ x^2-2x-49=0$
$x=1+5\sqrt{2}=8.07$ or $1-5\sqrt{2}=-6.07$.

49 $\sqrt{x^2+y^2+z^2}$

Projections of Lines

50 Yes; when the line is perpendicular to the plane.

51 When it is parallel to the plane.

52 When the plane of the triangle is parallel to the plane.

53 Yes. Let one side of the triangle be parallel to the plane.

54 Never.

Projections of Points

55 (0,4,5), (3,0,5), (3,4,0)

56 (0,0,4), (0,2,0)

57 Its projections fall on the two axes which are represented by coördinates other than zero.

58 The projections are the lines joining the following pairs of points: (0,3,5) and (0,6,7); (1,0,5) and (3,0,7); (1,3,0) and (3,6,0).

59 When the segment is perpendicular to one of the planes.

60 No. **61** No.

Measuring Projections

62 The end-points are (0,5,3) and (0,7,6); (2,0,3) and (1,0,6); (2,5,0) and (1,7,0). By $\sqrt{(x_1-x_2)^2+(y_2-y_1)^2+(z_2-z_1)^2}$, the lengths are $\sqrt{13}$, $\sqrt{10}$, and $\sqrt{5}$.

63 It is parallel to it.

64 A (0,7,5), B (0,1,9), C (0,6,8); A (2,0,5), B (4,0,9), C (3,0,8); A (2,7,0), B (4,1,0), C (3,6,0).

65 When plane of triangle is perpendicular to one of the coördinate planes.

66 When the plane of the triangle is parallel to one of the coördinate planes.

67 No.

68 $2\sqrt{13}$, $\sqrt{26}$, $\sqrt{10}$; $2\sqrt{5}$, $\sqrt{2}$, $\sqrt{10}$; $2\sqrt{10}$, $\sqrt{2}$, $\sqrt{26}$.

ANGLES FORMED BY PLANES

Dihedral Angles

69 90° **71** Parallel

70 Perpendicular **72** 180°

Loci

73 Pass planes bisecting angles formed by the walls, and by the floor and each wall. The intersections form a straight line, all of whose points satisfy the condition.

74 In the plane that bisects the line perpendicularly.

75 On a plane parallel to and midway between the floor and ceiling.

76 All such points lie on the line which is the intersection of planes parallel to and at the given distances from the floor and wall.

77 A circle with $r=12$, $C=24\pi$.

Polyhedral Angles

81 If the plane angles are equal and arranged in the same order, the polyhedral angles are equal.

82 It is impossible because the sum of the angles is greater than 360°.

SOLID FIGURES

Measuring Pyramids

86 1:9.

87 The lateral areas of similar solids are in the ratio of the squares of their sides.

88 $282.8.\left(S=\frac{nbh}{2}\text{ and }h=\sqrt{15^2-5^2}\right)$

89 A = lateral area of pyramid $=\frac{3\sqrt{3}}{4}\cdot 100$; truncated area $=A-\frac{A}{4}=97.4$.

Cubes

90 $6a^2=1350$

91 If a is the edge, by the Pythagorean Theorem, $a^2+a^2=100$, $a=\sqrt{50}$; $a^2+100=(\text{diagonal})^2$. Diagonal $=5\sqrt{6}$.

92 $a\sqrt{3}=3$; $a=\sqrt{3}$; $6a^2=18$. **94** 12

93 8 **95** 4

96 $a\cdot\sqrt{2}a=\sqrt{2}a^2$; $25\sqrt{2}$

97 $\sqrt{2}a^2=10$; $a^2=5\sqrt{2}$; $6a^2=30\sqrt{2}$

SOLID FIGURES (continued)

Prisms

98 Let x be an edge. $3x \cdot x = 12$; $x = 2$.

99 $3\sqrt{3}$, $3\sqrt{3}$, $3\sqrt{3}$

100 $A = 90 \cdot 50 + 2\sqrt{45 \cdot 5 \cdot 32 \cdot 8} = 4980$

101 If sides are $17x$, $10x$, $9x$, $S = \frac{1}{2}$ sum of sides $= 18x$, then total area is $32S + 2\sqrt{S(S-17x)(S-10x)(S-9x)} = 72x^2 + 576x = 1440$; $x = 2$ and sides are 34, 20, 18 in.

102 A parallelogram

Parallelepipeds

103 (a) 144 sq. in. (b) No.

104 (a) Yes. (b) No.

105 $2b^2 + 2b\sqrt{4a^2 - b^2}$. Draw the figure.

106 a, $\sqrt{a^2 + 2b^2}$, $\sqrt{a^2 + 4b^2}$. $b\sqrt{2a^2 - b^2}$, $ab\sqrt{2}$

107 Rectangles

108 Diagonals $\sqrt{189}$; sections $8\sqrt{125}$, $5\sqrt{164}$, $10\sqrt{89}$.

Cylinders

109 $32\pi \times 120\pi = 152\pi$

110 (a) Doubled; (b) tripled.

111 A circle

112 (a) Doubled; (b) quadrupled.

113 $r = 5$, $h = 5$; $A = 50\pi + 50\pi = 100\pi$.

Cones

114 $r = \dfrac{125.6}{2\pi}$, $l = \dfrac{125.6}{\pi}$, $s = \pi rl = 2512$ sq. in.

115 An isosceles triangle.

116 Doubled.

117 Tripled.

118 $\pi(rl + r^2)$

119 Quadrupled.

Areas of Surfaces

120 The long diagonal of the base is $2a$, a being a side of the base. Then $2ah = 1$; $S = ph = 6ah = 3$ sq. ft.

121 $\dfrac{S}{p} = h$; $\dfrac{45}{9} = 5$

122 For the area of the perpendicular section, apply $A = \sqrt{s(s-a)(s-b)(s-c)}$ where a, b, c are the sides of the triangle and $s = \dfrac{(a+b+c)}{2}$. $\sqrt{18 \cdot 9 \cdot 8 \cdot 1} = 36 = \frac{1}{4} \cdot 144$. The sides must be 18, 20, and 34, and $s = 8 \cdot (15 + 20 + 34) = 576$.

123 The area of the base is a^2. The base of the diagonal section (a triangle) is $a\sqrt{2}$. The area of the diagonal section is $\dfrac{ka\sqrt{2}}{2}$, where k is the altitude of the pyramid. $a^2 = \dfrac{ha\sqrt{2}}{2}$; $a\sqrt{2} = k$. By Pythagoras: $h^2 = \left(\dfrac{a}{2}\right)^2 + k^2$; $h^2 = \dfrac{a^2}{4} + 2a^2 = \dfrac{9a^2}{4}$; $h = \dfrac{3a}{2}$; $S = 4 \cdot \dfrac{3a}{2} \cdot \dfrac{a}{2} = 3a^2$.

124 Since the long diagonal of the base is twice a side, the altitude k of the pyramid is half the altitude h of a face. Since $h^2 - (a^2 - \frac{1}{4}a^2) = k^2 = \frac{1}{4}h^2$, $h = a$. Lateral area $= 3ah = 3a^2$.

125 First find area of trapezoids, each $\dfrac{\sqrt{3}}{6}(a^2 - b^2)$. Adding areas of bases (equilateral triangles), and reducing, $b = \dfrac{1}{3}\sqrt{27a^2 - 12S\sqrt{3}}$.

126 $S = \dfrac{\sqrt{2}}{4}\sqrt{C^2 - (A-B)^2}$

127 Total area: $S = \pi dh + \frac{1}{4}\pi d^2 \times 2 = \pi\left(2.66 + \dfrac{.49}{2}\right) = 2.905\pi$ sq. yds. $F = S \times P \times k$, where k is the number of sq. in. in 6 sq. yd. $F = 941.22\pi$ lbs. $=$ 2957 lbs.

128 S, the lateral area, $= \pi dh$, where h is the length. $\dfrac{4500}{550} = \dfrac{1}{36}\pi h$. $h = 93.8$ yd.

129 Area of a base $= \pi R^2$; lateral area $= 2\pi Rh$ $2\pi Rh = 2\pi R^2$; $\therefore h = R$.

130 Draw cross-section diagram. Slant height, $l, = \sqrt{4^2 + 3.5^2} = \frac{1}{2}\sqrt{113}$. Area $= \pi rl = 66.8$ sq. yds.

131 Hypotenuse is 13 in. long. Figure is two cones with equal bases, and slant heights of 5 in. and 12 in. Radius of base $r = \dfrac{5 \cdot 12}{13}$. Area $= \pi r(5 + 12) =$ 246 sq. in.

132 Base of cone is length of semicircle. Therefore, radius of cone is $\dfrac{r}{2}$.

133 $S = \pi(r_1 + r_2)l$, with $r_1 = 0.1$, $r_2 = 0.5$, $l = 0.7$ gives $S = 0.42\pi = 1.32$ sq. yd.

134 Radius of midsection: $\dfrac{\sqrt{\dfrac{A}{\pi}} + \sqrt{\dfrac{B}{\pi}}}{2}$. Area of midsection: $\dfrac{A + 2\sqrt{AB} + B}{4}$.

Volumes

135 $e^3 = 8$; $e = 2$; $6e^2 = 24$ sq. in.

136 $3^3 + 4^3 + 5^3 = 27 + 64 + 125 = 216$. $\sqrt[3]{216} = 6$ in.

137 Let $x =$ edge. $(x+2)^3 - x^3 = 98$; $x^2 + 2x - 15 = 0$; $x = -5$ or 3. Original cube had edge of 3 in.

138 $6e^2 = e^3$; $e = 6$.

139 $\sqrt{(2x)^2 + (3x)^2 + (6x)^2} = 35$; $x = 5$. Edges are 10, 15, 30. $V = 10 \times 15 \times 30 = 4500$ cu. in.

140 $h = \sqrt{10^2 - 9^2} = \sqrt{19}$. $V = \dfrac{42 + 24}{2} \times \sqrt{19} \times 5280 = 759{,}495$ cu. ft.

141 Cylinder has radius a and height a. $V = \pi r^2 h = \pi a^3$.

142 As the areas of their bases.

143 As their altitudes.

144 It should be multiplied by $\sqrt{2}$.

145 $\dfrac{a^3}{4\pi}$

146 $A\triangle = 2\sqrt{2}$; $h = 2\sqrt{2}$; $V = 8$; $e = 2$.

147 Diagonal of base $= 2\sqrt{5^2 - 3^2} = 8$. Area of base $= 32$ sq. in.; volume $= 32$ cu. in.

148 $S = 8 \cdot \dfrac{\sqrt{3}}{4}a^2 = 2\sqrt{3}a^2$; $V = 2 \cdot \frac{1}{3} \cdot a^2 \cdot \dfrac{a\sqrt{2}}{2} = \dfrac{\sqrt{2}}{3}a^3$.

149 $(\frac{1}{2})^3 = \frac{1}{8}$ of volume is cut off.

150 $S = \frac{1}{2} . 2\pi r \times 1 = \pi rl = 24\pi$. $rl = 24$; but $l = \sqrt{h^2 + r^2}$. $r\sqrt{36 + r^2} = 24$; $r^4 + 36r^2 - 576 = 0$; $r^2 = 12$; $V = \frac{1}{3}\pi r^2 h = 24\pi$

151 $V = \dfrac{c^2}{24\pi^2}\sqrt{4\pi^2 a^2 - c^2}$

152 $h = \dfrac{a}{2}$, $r = \dfrac{\sqrt{3}}{2}a$; $V = \frac{1}{3}\pi r^2 h = \dfrac{\pi a^3}{8}$.

153 52,500 cu. in.

154 The light falls on areas which vary as their linear dimensions squared. Their linear dimensions vary as their distance from the apex of the cone.

SPHERICAL SURFACES

Spheres

155 None, 1 (tangent), or 2.

156 1. **157** Yes

158 An infinite number.

159 An infinity. They form a cone.

160 An infinity. **161** 90°

162 With radius of cut $=R$, $R^2+a^2=r^2$; $R^2=r^2-a^2$; $A=\pi(r^2-a^2)$.

163 A circle. **164** Great circles.

Measuring Spheres

165 Radius desired is an altitude of $\triangle OAB$, where O is center of sphere and B any point 15 in. from A. The sides of this triangle are $12\frac{1}{2}$, $12\frac{1}{2}$, 15. Hence its area is $\sqrt{20\cdot7.5\cdot7.5\cdot5}=75$ and the altitude on the side $12\frac{1}{2}$ is 12 in.

166 Circle's radius is altitude (on hypotenuse) of rt. triangle whose sides are 25, 20, and 15, or 12; circumference $=75.4$ in.

167 $(36-x)^2+r^2=29^2$; $x^2+r^2=25^2$; $1296-72x=216$; $x=15$, $r=20$ in. Ans.: 125.7 in.

168 $S=225\pi$; $V=\frac{4}{3}\pi r^3=\frac{1125}{2}\pi$ cu. in.

169 $m\sqrt{m}=n\sqrt{n}$. **170** $\frac{4}{3}\pi r^3=4\pi r^2$; $r=3$.

171 Decreased $1:\frac{1}{6}\sqrt[3]{36}$

172 $\frac{V}{V^1}=\frac{m}{n}$; $\frac{r}{r^1}=\frac{\sqrt[3]{m}}{\sqrt[3]{n}}$; $\frac{S}{S^1}=\frac{\sqrt[3]{m^2}}{\sqrt[3]{n^2}}$

173 Area of circle $=\pi r^2=1$; of sphere $=4\pi r^2=4$.

Equations

174 $(x-1)^2+y^2+(z-1)^2=100$

175 $(y-5)^2+(z-3)^2=25$

176 $\left(\frac{300x}{299}\right)^2+y^2+z^2=r^2$

177 XOY-plane: $\frac{x^2}{3}+y^2=1$;

XOZ-plane: $\frac{x^2}{3}+\frac{5z^2}{12}=1$;

YOZ-plane; $y^2+\frac{5z^2}{12}=1$

These are ellipses.

IRRATIONALS AND IMAGINARIES

IRRATIONAL NUMBERS

Adding Radicals

1 $9\sqrt{5}$ **4** $5\sqrt{y}+5\sqrt{x}$

2 $9\sqrt{3}+2\sqrt{7}$ **5** $8\sqrt{a+b}$

3 $\sqrt{5}+5\sqrt{x}$ **6** $10\sqrt{c-d}+2\sqrt{c+d}$

Subtracting Radicals

7 $3\sqrt{3}$ **10** $4\sqrt{x}-6\sqrt{y}$

8 $\sqrt{5}$ **11** $3\sqrt{a+b}$

9 $2\sqrt{x}-3\sqrt{2}$ **12** $2\sqrt{a-b}-2\sqrt{a+b}$

Multiplying Radicals

13 42 **16** $72-25\sqrt{15}$

14 $12\sqrt{6}$ **17** $3a+2\sqrt{ab}-5b$

15 $8\sqrt{35}$ **18** $-42-15c+19\sqrt{7c}$

Dividing Radicals

19 $\sqrt{3}$ **22** $\sqrt{\frac{a}{b}}$ **25** $3\sqrt{3}$

20 $\sqrt{2}$ **23** $\sqrt{\frac{a+b}{a-b}}$ **26** $\frac{3\sqrt{3}}{2}$

21 $\sqrt{\frac{1}{25}}=\pm\frac{1}{5}$ **24** $\sqrt[3]{a^3}=a$ **27** $\sqrt[3]{b}$

Final Check

28 $7\sqrt{2}$ **34** $\frac{2\sqrt{3}}{3}$ **38** $\sqrt{3}+1$

29 $-10\sqrt{5}$

30 $25\sqrt{a}$ **35** $1\frac{1}{3}$ **39** $\frac{a\sqrt{a}+\sqrt{ab}+a\sqrt{b}+b}{a-b}$

31 $3a\sqrt{5}-6b$ **36** 1

32 $\frac{1}{6}\sqrt{30}$ **37** $\frac{5\sqrt{3}}{9}$ **40** $4+2\sqrt{5}+2\sqrt{3}+\sqrt{15}$

33 $\frac{1}{3}\sqrt[3]{9}$ **41** $\frac{11}{60}\sqrt{10}$

IMAGINARY NUMBERS

42 $2\sqrt{3}i$ **44** $3\sqrt{3}i$ **47** $65i$ **50** $\frac{1}{3}i$

43 $\sqrt{15}i$ **45** $6\sqrt{2}i$ **48** $50i$ **51** $\frac{1}{4}\sqrt{6}i$

46 $6\sqrt{2}i$ **49** $6\sqrt{15}i$

Combining Imaginaries

52 $(2+\sqrt{3})i$ **55** -18 **57** $-2\sqrt{5}i$

53 i

54 $\frac{8}{5}$ **56** $\frac{\sqrt{7}}{7}$ **58** $\frac{2}{i}=-2i$

Complex Numbers

59 $5+21i$ **64** $6-6\sqrt{6}+(6\sqrt{2}+6\sqrt{3})i$

60 $7+15i$

61 $4-4i$ **65** $\frac{6+6\sqrt{6}+(6\sqrt{2}-6\sqrt{3})i}{36}$

62 $3+(\sqrt{5}+\sqrt{3})i$

63 $-55+46i$ **66** $\frac{36+9\sqrt{3}i}{19}$

MEASURING ROD

1 The cone is a revolved equilateral triangle and the sphere will be the revolved inscribed circle. Bisectors meet at point $\sqrt{3}$ in. from each face. Radius of sphere $=1.732$ in.

2 $\pi r^2h=V$; $(3.14)(0.325)^2h=1$; $h=3.015$ in.

3 Construct perpendicular radius and a radius to one end of the chord. Then $x^2+(\frac{1}{2})^2=4$; $x=\frac{1}{2}\sqrt{55}=1.854'$. Ht. $=3.854'$.

4 $2.5\pi\sqrt{42.25}=51.051$ sq. ft.

5 $100\cdot\pi\cdot4^2\cdot1.10\cdot\frac{1}{144}=38.4$ sq. ft.

6 56.55 sq. ft.

7 $\frac{2\times18\times16+2\times18\times9+2\times16\times9}{144}\times10=$ 82.5 sq. ft.

8 $\frac{1.5\pi\sqrt{38.25}\times12\times1.15}{144}=2.79$ sq. ft.

9 $12\cdot1.10\frac{\pi(5+4)\sqrt{18^2+1^2}+\pi\cdot4^2}{144}=51.3$ sq. ft.

10 Perimeter $=2\pi r=12$; area $=2\pi r^2=23$ sq. ft.

11 Area of roof is $32\times80\times2=5120$ sq. ft.

12 $A=\frac{2\pi r^2+2\pi rh+5}{144}\times1000=340$ sq. ft.

13 $36\pi=113.1$ sq. ft.

14 Area of trapezoidal section, $16\frac{3+7}{2}$; volume, $80\cdot40=3200$ cu. ft.

15 96 cu. in. of copper in the bar. $V=96=l\cdot\pi\cdot(\frac{1}{32})^2\div12=2608$ ft.

16 $V=4\times60\times20+\frac{1}{2}\times7\times60\times20=9000$ cu. ft.

17 One rail: $2\cdot5280\cdot\frac{5}{144}=366.7$ cu. ft. $\sqrt[3]{366.7}=$ 7.16 ft. cube.

Two rails: $\sqrt[3]{733.3}=9.02$ ft. cube

18 $\frac{4}{3}\pi r^3=24\pi$; $r=\sqrt[3]{18}$; $4\pi r^2=12\pi\sqrt[3]{12}=86.3$ sq. in.

MEASURING ROD
(continued)

19 Weight of sphere $=0.75\cdot\frac{4}{3}\pi r^3=\pi r^3$. Volume ($=$weight) of water displaced, $\frac{4}{3}\pi r^3-\frac{1}{3}h^2(3r-h)$ where $h=$height above water. Equating, $r^3=\frac{4}{3}r^3-\frac{1}{3}(rh^2-h^3)$, or, putting $\frac{h}{r}=x$, $x^3-3x^2+1=0$, and (solving by trial) $x=0.653$.

20 A small cone at the vertex of the original cone is removed in the drilling. Total volume removed is $\left(5+\frac{\sqrt{3}}{3}\right)\pi=17.51$ cu. in.

21 $8\div\sqrt[3]{2}=6.35$

22 Area of circle $=\pi$. Area of hexagon $=6\frac{\sqrt{3}}{4}$. % decrease in area of base $=$ % decrease in volume $=\frac{\pi-1.5\sqrt{3}}{\pi}\cdot 100\%=17.3\%$

23 If $q=$radius of base of cone, $r=$radius of sphere, $h_1=$distance, center of sphere to base of cone, then volume of cone $=\frac{1}{3}\pi q^2(h_1+r)=\frac{1}{3}\pi q^2(r+\sqrt{r^2-q^2})$. Substituting $q=1.5$, $r=4$, volume of cone $=18.2$ cu. in.

24 $\pi\frac{\sqrt{3}}{3}=1.81$ cu. in.

25 Volume $=\frac{4}{3}\pi r^3-\frac{4}{3}\pi(r-\frac{1}{4})^3=108.6$ cu. in.

26 Radius of drill $\frac{1}{2}$, altitude of drill $\sqrt{15}$, volume of drill $\frac{\pi}{4}\sqrt{15}$; radius of sphere 2, volume of sphere $\frac{32}{3}\pi$; weight of drill $\frac{3}{32}\cdot\frac{\sqrt{15}}{4}\cdot 11=1$ pound.

27 Formula for volume of spherical segment of height h, $\frac{\pi}{3}h^2(3r-h)$; volume of sphere $=\frac{4\pi}{3}r^3$. Hence we have $\frac{h^2(3r-h)}{r^3}=\left(\frac{h}{r}\right)^2\left(3-\frac{h}{r}\right)=\frac{3}{10}$.
Solving (by trial) $\frac{h}{r}=0.335$, $1-\frac{h}{r}=0.665$, and the cut must be made $\frac{2}{3}$ the radius from the center.

28 Ratio of volumes 4:1; ratio of lateral areas 2:1.

29 Volume of cone, $\frac{1}{3}\pi(\frac{3}{2})^2\cdot 4=3\pi$. Volume of sphere $=\frac{\pi}{6}\cdot(\frac{1}{8})^3$. Ratio $=3\cdot 6\cdot 8^3=9216$ shot.

30 $14\sqrt[3]{2}=17.6$ in.

31 Using formula, $\frac{h}{6}(B_1+4B_m+B_2)$, we get volume $=\frac{\pi}{6}\left[\left(\frac{20}{2}\right)^2+4\cdot\left(\frac{25}{2}\right)^2+\left(\frac{21}{2}\right)^2\right]\cdot\frac{42}{1728}=$ 10.63 cu. ft.

32 Formula, $\frac{1}{3}\pi h(r_1^2+r_1r_2+r_2^2)$, gives volume, $\pi\left(\frac{1}{4}+\frac{5}{32}+\frac{25}{256}\right)=\frac{129\pi}{256}$. Weight $=0.26\cdot\frac{129\pi}{256}=$ 0.412 lb.

33 Volume of tubing $=15\pi(1\frac{1}{4}^2-1^2)$; of shell $=\frac{5}{2}\pi\cdot\frac{9}{16}$; wt. of shell, $0.28\cdot\frac{45}{32}\pi=1.24$ lb.

34 Volume of hemisphere, $\frac{2}{3}\pi\cdot 5^3$; of cylinder, $\pi\cdot 5^2\cdot 11$; total volume, $5^2\pi\cdot 14\cdot 33=1126$ cu. in.

35 $V=\frac{h}{3}(B_1+B_2+\sqrt{B_1B_2})=96+140+\sqrt{13440}=$ 352 cu. ft. $\frac{352}{27}=13.0$ cu. yd.

36 $\frac{24\cdot 36\cdot 11}{2\cdot 231}=\frac{12\cdot 36}{21}=\frac{144}{7}=20\frac{4}{7}$ gal.

37 $V=\frac{1}{3}\times\left(6\times\frac{1}{3}\times\frac{\sqrt{3}}{3}\right)\times 4=1.54$ cu. ft.

38 Area of annulus $=\frac{\pi}{4}(4^2-3.5^2)=\frac{\pi}{4}\cdot\frac{15}{4}$
Volume of 300 $=300\cdot\frac{1}{2}\cdot\frac{15\pi}{16}=\frac{4500\pi}{32}=442$ cu. in.

39 Volume of tetrahedron $=\frac{1}{12}\sqrt{2}l^3=40.1$ cu. in.

40 Volume $=8\cdot 72=576$ cu. in. $=\frac{\sqrt{2}}{12}l^3$ and $l=$ 16.97 in.

41 Volume of cylinder, 16π; volume of frustum $\frac{5}{3}\pi(3^2+3b+b^2)$ where b is upper base. Hence $b^2+3b+9=9.6$ and $b=0.19$ in.

42 $V=9\times 6\times\pi\times 6\times 1728$ cu. in.
$\frac{9\times 6\times 6\times 1728\times\pi}{231}=7614$ gal.

43 Assuming a conical form, volume $=\frac{7\pi\cdot 11^2}{3\cdot 4\cdot 27}=$ 8.2 cu. yd.

44 Wt. $=\frac{1}{3}\cdot 764^2\cdot 480\cdot 200=18.68\cdot 10^9$ lb.

45 $365\cdot 1\cdot\frac{1}{24}\cdot\frac{1}{48}\ 3000=950.5$ lb.

46 Wt. $=\frac{1}{4}\cdot 5\times 100\times 0.26=32.5$ lb.

47 $\frac{3}{4}\times\pi\times\frac{121}{4}\times 7=500$ cu. ft.

48 Volume of wheel $=\frac{3}{4}\pi\cdot 4^2=12\pi$; of flange $=2\pi(5^2-4^2)=18\pi$; total volume $=30\pi=94.2$ cu. in.

49 1728 cu. in. $=10^8d$ where $d=$thickness. $d=1.73\cdot 10^{-5}=0.0000173$ in.

50 $\frac{1}{6}\pi d^3=216$; $d=7.44$ in.

51 Cross section of stream, 60° sector of circle, area $\left(\frac{\pi}{6}-\frac{\sqrt{3}}{4}\right)\cdot 20^2$. Velocity of stream, 4.4 ft. per sec. Volume per sec., 159 cu. ft.

52 Wt. $=10\cdot 200(9\cdot 16-3[4\cdot 5+\frac{1}{2}\pi\cdot 2^2])=130{,}300$ lb.

53 Let O be the near vertex of the cube, and P the point of projection, so that $OP=8$. Take the three near edges of the cube (produced) as three coördinate axes, so that the far corner of the cube is the point $(-6,\ -6,\ -6)$. The length to the diagonal is $6\sqrt{3}$ and P lies on the diagonal produced, hence the coördinates of P are $\frac{8}{\sqrt{3}}$, $\frac{8}{\sqrt{3}}$, $\frac{8}{\sqrt{3}}$. Let C be one of the three vertices nearest to O. Then the coördinates of C are $0,0,-6$. The distance PC is found from $\overline{PC}^2=\frac{64}{3}+\frac{64}{3}+\left(6+\frac{8}{\sqrt{3}}\right)^2=100+\frac{96}{\sqrt{3}}$. If B be one of the three next nearest vertices, the coördinates of B are $0,-6,-6$ and PB is found from $\overline{PB}^2=\frac{64}{3}+\left(6+\frac{8}{\sqrt{3}}\right)^2+\left(6+\frac{8}{\sqrt{3}}\right)^2=136+\frac{192}{\sqrt{3}}$. Thus we can construct the triangle OPC ($OP=8$, $OC=6$); we can also construct the triangle OPB ($OP=8$, $OB=6\sqrt{2}$). On OP, erect a $\perp$ at some point P', meeting PC at C' and another $\perp$ meeting PB at B'. In the desired projection, if p is the projection of O, and b and c of B and C, then the distance $pb=P'B'$, $pc=P'C'$. The three near edges will project into three lines pa, pc, and pd equal in length to pc and making angles of 120° with each other; and the three next nearest vertices, into points b, e, and f, distant pb from p. The eighth vertex is invisible in the projection and lies directly behind p.

55 $[x-(2+3i)][x-(2-3i)]=x^2-4x+13=0$; $x^2-6x+13=0$

56 $x=\frac{-5\pm\sqrt{25-56}}{4}=\frac{-5\pm\sqrt{31}i}{4}$

57 Let the paper be at (0,0,0). The light is then at (4,3,12). Applying $\sqrt{x^2+y^2+z^2}$, we get the distance as 13 feet.

58 $100=\sqrt{50^2+50^2+z^2}$; $z^2=5000$; $z=50\sqrt{2}=70.7'$
$70.7'+5'=75.7'$

59 $d=$diam., $h=$ht. of cone. Then $\frac{1}{12}\pi d^3=\frac{1}{3}\cdot\frac{1}{4}\pi d^2h$ or $h=d$, $2\frac{1}{2}$ in.

60 $V=\frac{1}{2}e^3=\frac{1}{2}$ cu. ft.

Tables and Formulas

TABLE XXXI
EQUIVALENT MEASUREMENTS OF ANGLES

1 revolution = 4 right angles
1 revolution = 360°
1 revolution = 21,600′
1 revolution = 1,296,000″
1 revolution = 2π radians
1 right angle = 1600 mils
1 radian = 57.29578°
1 radian = 57°17′44.8″
1 radian = 1018.59 mils
1 degree = 0.01745329 radian
1 minute = 0.000290888 radian
1 second = 0.0000048481 radian
1 mil = 0.000982 radian

TABLE XXXII
RIGHT ANGLES EXPRESSED IN DEGREES AND RADIANS

$$2 \text{ right angles} = 180° = \pi \text{ radians}$$

$$1 \text{ right angle} = 90° = \frac{\pi}{2} \text{ radians}$$

$$\frac{2}{3} \text{ right angle} = 60° = \frac{\pi}{3} \text{ radians}$$

$$\frac{1}{2} \text{ right angle} = 45° = \frac{\pi}{4} \text{ radians}$$

$$\frac{1}{3} \text{ right angle} = 30° = \frac{\pi}{6} \text{ radians}$$

TABLE XXXIII
DEGREES AND MINUTES EXPRESSED IN RADIANS

Degrees	Radians	Minutes	Radians
1	0.01745	1	0.00029
2	0.03491	2	0.00058
3	0.05236	3	0.00087
4	0.06981	4	0.00116
5	0.08727	5	0.00145
6	0.10472	6	0.00175
7	0.12217	7	0.00204
8	0.13963	8	0.00233
9	0.15708	9	0.00262
10	0.17453	10	0.00291
20	0.34907	20	0.00582
30	0.52360	30	0.00873
40	0.69813	40	0.01164
50	0.87266	50	0.01454

TABLE XXXIV
SIGNS OF THE TRIGONOMETRIC FUNCTIONS

	θ in Quadrant I	II	III	IV
$\sin\theta$	+	+	−	−
$\cos\theta$	+	−	−	+
$\tan\theta$	+	−	+	−
$\cot\theta$	+	−	+	−
$\sec\theta$	+	−	−	+
$\csc\theta$	+	+	−	−

TABLE XXXV
VALUES OF THE TRIGONOMETRIC FUNCTIONS

θ AS MULTIPLE OF 90°

	0° or 360°	90°	180°	270°
$\sin\theta$	0	1	0	−1
$\cos\theta$	1	0	−1	0
$\tan\theta$	0	...	0	...
$\cot\theta$	...	0	...	0
$\sec\theta$	1	...	−1	...
$\csc\theta$	...	1	...	−1

TABLE XXXVI
VALUES OF THE TRIGONOMETRIC FUNCTIONS

$\theta \pm$ MULTIPLES OF 90° IN TERMS OF θ

	θ	$-\theta$	$90°+\theta$	$90°-\theta$	$180°+\theta$	$180°-\theta$	$270°+\theta$	$270°-\theta$
sin	$\sin\theta$	$-\sin\theta$	$\cos\theta$	$\cos\theta$	$-\sin\theta$	$\sin\theta$	$-\cos\theta$	$-\cos\theta$
cos	$\cos\theta$	$\cos\theta$	$-\sin\theta$	$\sin\theta$	$-\cos\theta$	$-\cos\theta$	$\sin\theta$	$-\sin\theta$
tan	$\tan\theta$	$-\tan\theta$	$-\cot\theta$	$\cot\theta$	$\tan\theta$	$-\tan\theta$	$-\cot\theta$	$\tan\theta$

TABLE XXXVII
VALUES OF THE TRIGONOMETRIC FUNCTIONS

θ AT SELECTED INTERVALS

	−30°	30°	45°	60°	120°	225°
$\sin\theta$	$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}\sqrt{2}$	$\frac{1}{2}\sqrt{3}$	$\frac{1}{2}\sqrt{3}$	$-\frac{1}{2}\sqrt{2}$
$\cos\theta$	$\frac{1}{2}\sqrt{3}$	$\frac{1}{2}\sqrt{3}$	$\frac{1}{2}\sqrt{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}\sqrt{2}$
$\tan\theta$	$-\frac{1}{3}\sqrt{3}$	$\frac{1}{3}\sqrt{3}$	1	$\sqrt{3}$	$-\sqrt{3}$	1
$\cot\theta$	$-\sqrt{3}$	$\sqrt{3}$	1	$\frac{1}{3}\sqrt{3}$	$-\frac{1}{3}\sqrt{3}$	1
$\sec\theta$	$\frac{2}{3}\sqrt{3}$	$\frac{2}{3}\sqrt{3}$	$\sqrt{2}$	2	−2	$-\sqrt{2}$
$\csc\theta$	−2	2	$\sqrt{2}$	$\frac{2}{3}\sqrt{3}$	$\frac{2}{3}\sqrt{3}$	$-\sqrt{2}$

TABLE XXXVIII
GREEK ALPHABET

Caps	Lower Case	Greek Name	Caps	Lower Case	Greek Name	Caps	Lower Case	Greek Name
Α	α	Alpha.	Ι	ι	Iota.	Ρ	ρ	Rho.
Β	β	Beta.	Κ	κ	Kappa.	Σ	σ ς	Sigma.
Γ	γ	Gamma.	Λ	λ	Lambda.	Τ	τ	Tau.
Δ	δ ∂	Delta.	Μ	μ	Mu.	Υ	υ	Upsilon.
Ε	ε	Epsilon.	Ν	ν	Nu.	Φ	ϕ φ	Phi.
Ζ	ζ	Zeta.	Ξ	ξ	Xi.	Χ	χ	Chi.
Η	η	Eta.	Ο	ο	Omicron.	Ψ	ψ	Psi.
Θ	θ ϑ	Theta.	Π	π	Pi.	Ω	ω	Omega.

TABLE XXXIX

VALUES OF TRIGONOMETRIC FUNCTIONS

(For angles over 45°, use the numbers on the right and the functions below.)

Read Down	sin	cos	tan	cot	Read Up
0°	.00000	1.00000	.00000		90°
15′	.00436	.99999	.00436	229.1820	45′
30′	.00872	.99996	.00873	114.5890	30′
45′	.01309	.99991	.01309	76.3900	15′
1°	.01745	.99985	.01745	57.2900	89°
15′	.02181	.99976	.02182	45.8294	45′
30′	.02617	.99966	.02619	38.1885	30′
45′	.03053	.99953	.03055	32.7303	15′
2°	.03489	.99939	.03492	28.6363	88°
15′	.03926	.99923	.03929	25.4517	45′
30′	.04361	.99905	.04366	22.9038	30′
45′	.04797	.99885	.04803	20.8188	15′
3°	.05234	.99863	.05241	19.0811	87°
15′	.05669	.99839	.05678	17.6106	45′
30′	.06105	.99814	.06116	16.3499	30′
45′	.06540	.99786	.06554	15.2571	15′
4°	.06976	.99756	.06993	14.3007	86°
15′	.07411	.99725	.07431	13.4566	45′
30′	.07846	.99692	.07870	12.7062	30′
45′	.08281	.99657	.08309	12.0346	15′
5°	.08716	.99620	.08749	11.4301	85°
15′	.09150	.99581	.09188	10.8829	45′
30′	.09585	.99540	.09629	10.3854	30′
45′	.10019	.99497	.10070	9.93101	15′
6°	.10453	.99452	.10510	9.51436	84°
15′	.10887	.99406	.10952	9.13093	45′
30′	.11320	.99357	.11394	8.77689	30′
45′	.11754	.99307	.11836	8.44896	15′
7°	.12187	.99255	.12278	8.14435	83°
15′	.12620	.99201	.12722	7.86064	45′
30′	.13053	.99145	.13165	7.59575	30′
45′	.13485	.99087	.13609	7.34786	15′
8°	.13917	.99026	.14054	7.11537	82°
15′	.14349	.98965	.14499	6.89688	45′
30′	.14781	.98902	.14945	6.69116	30′
45′	.15212	.98836	.15392	6.49710	15′
9°	.15643	.98769	.15838	6.31375	81°
15′	.16074	.98699	.16286	6.14023	45′
30′	.16505	.98629	.16734	5.97576	30′
45′	.16935	.98556	.17183	5.81966	15′
10°	.17365	.98481	.17633	5.67128	80°
15′	.17794	.98404	.18083	5.53007	45′
30′	.18224	.98326	.18534	5.39552	30′
45′	.18652	.98245	.18985	5.26715	15′
11°	.19081	.98163	.19438	5.14455	79°
15′	.19509	.98079	.19891	5.02734	45′
	cos	sin	cot	tan	

Read Down	sin	cos	tan	cot	Read Up
11°30′	.19937	.97993	.20345	4.91516	78° 30′
45′	.20364	.97905	.20800	4.80769	15′
12°	.20791	.97815	.21255	4.70463	78°
15′	.21218	.97723	.21712	4.60572	45′
30′	.21644	.97630	.22170	4.51071	30′
45′	.22070	.97534	.22628	4.41936	15′
13°	.22495	.97437	.23087	4.33148	77°
15′	.22920	.97338	.23547	4.24685	45′
30′	.23345	.97237	.24008	4.16530	30′
45′	.23768	.97134	.24469	4.08666	15′
14°	.24192	.97030	.24933	4.01078	76°
15′	.24615	.96923	.25397	3.93751	45′
30′	.25038	.96815	.25862	3.86671	30′
45′	.25460	.96705	.26328	3.79827	15′
15°	.25882	.96593	.26795	3.73205	75°
15′	.26303	.96478	.27263	3.66796	45′
30′	.26724	.96363	.27733	3.60588	30′
45′	.27144	.96245	.28203	3.54573	15′
16°	.27564	.96126	.28675	3.48741	74°
15′	.27983	.96005	.29147	3.43084	45′
30′	.28402	.95882	.29621	3.37594	30′
45′	.28820	.95757	.30096	3.32264	15′
17°	.29237	.95631	.30573	3.27085	73°
15′	.29654	.95502	.31051	3.22053	45′
30′	.30071	.95372	.31529	3.17159	30′
45′	.30486	.95240	.32010	3.12400	15′
18°	.30902	.95106	.32492	3.07768	72°
15′	.31316	.94970	.32975	3.03260	45′
30′	.31731	.94832	.33468	2.98868	30′
45′	.32144	.94693	.33945	2.94591	15′
19°	.32557	.94552	.34433	2.90421	71°
15′	.32969	.94409	.34922	2.86356	45′
30′	.33381	.94264	.35412	2.82391	30′
45′	.33792	.94118	.35904	2.78523	15′
20°	.34202	.93969	.36397	2.74748	70°
15′	.34612	.93819	.36892	2.71062	45′
30′	.35021	.93667	.37388	2.67462	30′
45′	.35429	.93514	.37887	2.63945	15′
21°	.35436	.93358	.38386	2.60509	69°
15′	.36244	.93201	.38888	2.57150	45′
30′	.36650	.93042	.39391	2.53865	30′
45′	.37056	.92881	.39896	2.50652	15′
22°	.37461	.92718	.40403	2.47509	68°
15′	.37865	.92554	.40911	2.44433	45′
30′	.38268	.92388	.41422	2.41421	30′
45′	.38671	.92220	.41934	2.38473	67° 15′
	cos	sin	cot	tan	

TABLE XXXIX (continued)

VALUES OF TRIGONOMETRIC FUNCTIONS

(For angles over 45°, use the numbers on the right and the functions below.)

Read Down	sin	cos	tan	cot	Read Up
23°	.39073	.92051	.42448	2.35585	**67°**
15′	.39474	.91879	.42963	2.32756	**45′**
30′	.39875	.91706	.43481	2.29984	**30′**
45′	.40275	.91531	.44001	2.27267	**15′**
24°	.40674	.91355	.44523	2.24604	**66°**
15′	.41072	.91176	.45047	2.21992	**45′**
30′	.41469	.90996	.45573	2.19430	**30′**
45′	.41866	.90814	.46101	2.16917	**15′**
25°	.42262	.90631	.46631	2.14451	**65°**
15′	.42657	.90446	.47163	2.12030	**45′**
30′	.43051	.90259	.47697	2.09654	**30′**
45′	.43445	.90070	.48234	2.07321	**15′**
26°	.43837	.89879	.48773	2.05030	**64°**
15′	.44229	.89687	.49315	2.02780	**45′**
30′	.44620	.89493	.49858	2.00569	**30′**
45′	.45010	.89298	.50404	1.98396	**15′**
27°	.45399	.89101	.50953	1.96261	**63°**
15′	.45787	.88902	.51503	1.94162	**45′**
30′	.46175	.88701	.52057	1.92098	**30′**
45′	.46562	.88499	.52613	1.90069	**15′**
28°	.46947	.88295	.53171	1.88073	**62°**
15′	.47332	.88089	.53732	1.86109	**45′**
30′	.47716	.87882	.54296	1.84177	**30′**
45′	.48099	.87673	.54862	1.82276	**15′**
29°	.48481	.87462	.55431	1,80405	**61°**
15′	.48862	.87249	.56003	1.78563	**45′**
30′	.49242	.87036	.56577	1.76749	**30′**
45′	.49622	.86820	.57155	1.74964	**15′**
30°	.50000	.86603	.57735	1.73205	**60°**
15′	.50377	.86384	.58318	1.71473	**45′**
30′	.50754	.86163	.58905	1.69766	**30′**
45′	.51129	.85941	.59494	1.68085	**15′**
31°	.51504	.85717	.60086	1.66428	**59°**
15′	.51877	.85491	.60682	1.64795	**45′**
30′	.52250	.85264	.61280	1.63185	**30′**
45′	.52621	.85035	.61882	1.61598	**15′**
32°	.52992	.84805	.62487	1.60033	**58°**
15′	.53362	.84573	.63095	1.58490	**45′**
30′	.53730	.84339	.63707	1.56969	**30′**
45′	.54097	.84104	.64322	1.55467	**15′**
33°	.54464	.83867	.64941	1.53986	**57°**
15′	.54829	.83629	.65563	1,52525	**45′**
30′	.55194	.83389	.66189	1.51084	**30′**
45′	.55557	.83147	.66818	1.49661	**56° 15′**
	cos	**sin**	**cot**	**tan**	

Read Down	sin	cos	tan	cot	Read Up
34°	.55919	.82904	.67451	1.48256	**56°**
15′	.56281	.82659	.68088	1.46870	**45′**
30′	.56641	.82413	.68728	1.45501	**30′**
45′	.56999	.82165	.69373	1.44149	**15′**
35°	.57358	.81915	.70021	1.42815	**55°**
15′	.57715	.81664	.70673	1.41497	**45′**
30′	.58070	.81412	.71329	1.40195	**30′**
45′	.58425	.81157	.71990	1.38909	**15′**
36°	.58779	.80902	.72654	1.37638	**54°**
15′	.59131	.80645	.73323	1.36383	**45′**
30′	.59482	.80385	.73996	1.35142	**30′**
45′	.59833	.80125	.74674	1.33916	**15′**
37°	.60182	.79864	.75356	1.32704	**53°**
15′	.60529	.79600	.76042	1.31507	**45′**
30′	.60876	.79335	.76733	1.30323	**30′**
45′	.61222	.79069	.77429	1.29152	**15′**
38°	.61566	.78801	.78129	1.27994	**52°**
15′	.61909	.78532	.78834	1.26849	**45′**
30′	.62252	.78261	.79544	1.25717	**30′**
45′	.62592	.77989	.80259	1.24597	**15′**
39°	.62932	.77715	.80978	1.23490	**51°**
15′	.63271	.77439	.81703	1.22394	**45′**
30′	.63608	.77163	.82434	1.21310	**30′**
45′	.63944	.76884	.83169	1.20237	**15′**
40°	.64279	.76605	.83910	1.19175	**50°**
15′	.64613	.76323	.84656	1.18125	**45′**
30′	.64945	.76041	.85408	1.17085	**30′**
45′	.65276	.75757	.86166	1.16056	**15′**
41°	.65606	.75471	.86928	1.15038	**49°**
15′	.65935	.75184	.87698	1.14028	**45′**
30′	.66262	.74896	.88473	1.13029	**30′**
45′	.66588	.74606	.89253	1.12041	**15′**
42°	.66913	.74315	.90040	1.11061	**48°**
15′	.67237	.74022	.90834	1.10091	**45′**
30′	.67559	.73727	.91633	1.09131	**30′**
45′	.67880	.73432	.92439	1.08179	**15′**
43°	.68200	.73135	.93252	1.07237	**47°**
15′	.68518	.72837	.94071	1.06303	**45′**
30′	.68836	.72537	.94897	1.05378	**30′**
45′	.69151	.72236	.95729	1.04461	**15′**
44°	.69466	.71934	.96569	1.03553	**46°**
15′	.69779	.71630	.97415	1.02653	**45′**
30′	.70091	.71325	.98269	1.01761	**30′**
45′	.70402	.71019	.99131	1.00876	**15′**
45°	.70711	.70711	1.00000	1.00000	**45°**
	cos	**sin**	**cot**	**tan**	

TABLE XL

LOGARITHMS OF TRIGONOMETRIC FUNCTIONS

(10 is to be subtracted from the number in the table.)
(For angles over 45°, use the numbers on the right and the functions below.)

Read down	log sin	log cos	log tan	log cot	Read up
0°	....	0.00000	....	....	90°
15′	7.63982	0.00000	7.63982	2.36018	45′
30′	7.94084	9.99998	7.94086	2.05914	30′
45′	8.11693	9.99996	8.11696	1.88304	15′
1°	8.24186	9.99993	8.24192	1.75808	89°
15′	8.33875	9.99990	8.33886	1.66114	45′
30′	8.41792	9.99985	8.41807	1.58193	30′
45′	8.48485	9.99980	8.48505	1.51495	15′
2°	8.54282	9.99974	8.54308	1.45692	88°
15′	8.59395	9.99967	8.59428	1.40572	45′
30′	8.63968	9.99959	8.64009	1.35991	30′
45′	8.68104	9.99950	8.68154	1.31846	15′
3°	8.71880	9.99940	8.71940	1.28060	87°
15′	8.75353	9.99930	8.75423	1.24577	45′
30′	8.78568	9.99919	8.78649	1.21351	30′
45′	8.81560	9.99907	8.81653	1.18347	15′
4°	8.84358	9.99894	8.84464	1.15536	86°
15′	8.86987	9.99880	8.87106	1.12894	45′
30′	8.89464	9.99866	8.89598	1.10402	30′
45′	8.91807	9.99851	8.91957	1.08043	15′
5°	8.94030	9.99834	8.94195	1.05805	85°
15′	8.96143	9.99817	8.96325	1.03675	45′
30′	8.98157	9.99800	8.98358	1.01642	30′
45′	9.00082	9.99781	9.00301	0.99699	15′
6°	9.01923	9.99761	9.02162	0.97838	84°
15′	9.03690	9.99741	9.03948	0.96052	45′
30′	9.05386	9.99720	9.05666	0.94334	30′
45′	9.07018	9.99698	9.07320	0.92680	15′
7°	9.08589	9.99675	9.08914	0.91086	83°
15′	9.10106	9.99651	9.10454	0.89546	45′
30′	9.11570	9.99627	9.11943	0.88057	30′
45′	9.12985	9.99601	9.13384	0.86616	15′
8°	9.14356	9.99575	9.14780	0.85220	82°
15′	9.15683	9.99548	9.16135	0.83865	45′
30′	9.16970	9.99520	9.17450	0.82550	30′
45′	9.18220	9.99492	9.18728	0.81272	15′
9°	9.19433	9.99462	9.19971	0.80029	81°
15′	9.20613	9.99432	9.21182	0.78818	45′
30′	9.21761	9.99400	9.22361	0.77639	30′
45′	9.22878	9.99368	9.23510	0.76490	15′
10°	9.23967	9.99335	9.24632	0.75368	80°
15′	9.25028	9.99301	9.25727	0.74273	45′
30′	9.26063	9.99267	9.26797	0.73203	30′
45′	9.27073	9.99231	9.27842	0.72158	79° 15′
	log cos	log sin	log cot	log tan	

TABLE XL (continued)

LOGARITHMS OF TRIGONOMETRIC FUNCTIONS

(10 is to be subtracted from the number in the table.)
(For angles over 45°, use the numbers on the right and the functions below.)

Read down	log sin	log cos	log tan	log cot	Read up
11°	9.28060	9.99195	9.28865	0.71135	79°
15′	9.29024	9.99157	9.29866	0.70134	45′
30′	9.29966	9.99119	9.30846	0.69154	30′
45′	9.30887	9.99080	9.31806	0.68194	15′
12°	9.31788	9.99040	9.32747	0.67253	78°
15′	9.32670	9.99000	9.33670	0.66330	45′
30′	9.33534	9.98958	9.34576	0.65424	30′
45′	9.34380	9.98916	9.35464	0.64536	15′
13°	9.35209	9.98872	9.36336	0.63664	77°
15′	9.36022	9.98828	9.37193	0.62807	45′
30′	9.36819	9.98783	9.38035	0.61965	30′
45′	9.37600	9.98737	9.38863	0.61137	15′
14°	9.38368	9.98690	9.39677	0.60323	76°
15′	9.39121	9.98643	9.40478	0.59522	45′
30′	9.39860	9.98594	9.41266	0.58734	30′
45′	9.40586	9.98545	9.42041	0.57959	15′
15°	9.41300	9.98494	9.42805	0.57195	75°
15′	9.42001	9.98443	9.43558	0.56442	45′
30′	9.42690	9.98391	9.44299	0.55701	30′
45′	9.43367	9.98338	9.45029	0.54971	15′
16°	9.44034	9.98284	9.45750	0.54250	74°
15′	9.44689	9.98229	9.46460	0.53540	45′
30′	9.45334	9.98174	9.47160	0.52840	30′
45′	9.45969	9.98117	9.47852	0.52148	15′
17°	9.46594	9.98060	9.48534	0.51466	73°
15′	9.47209	9.98001	9.49207	0.50793	45′
30′	9.47814	9.97942	9.49872	0.50128	30′
45′	9.48411	9.97882	9.50529	0.49471	15′
18°	9.48998	9.97821	9.51178	0.48822	72°
15′	9.49577	9.97759	9.51819	0.48181	45′
30′	9.50148	9.97696	9.52452	0.47548	30′
45′	9.50710	9.97632	9.53078	0.46922	15′
19°	9.51264	9.97567	9.53697	0.46303	71°
15′	9.51811	9.97501	9.54309	0.45691	45′
30′	9.52350	9.97435	9.54915	0.45085	30′
45′	9.52881	9.97367	9.55514	0.44486	15′
20°	9.53405	9.97299	9.56107	0.43893	70°
15′	9.53922	9.97229	9.56693	0.43307	45′
30′	9.54433	9.97159	9.57274	0.42726	30′
45′	9.54936	9.97087	9.57849	0.42151	15′
21°	9.55433	9.97015	9.58418	0.41582	69°
15′	9.55923	9.96942	9.58981	0.41019	45′
30′	9.56408	9.96868	9.59540	0.40460	30′
45′	9.56886	9.96793	9.60093	0.39907	15′
22°	9.57358	9.96717	9.60641	0.39359	68°
15′	9.57824	9.96640	9.61184	0.38816	45′
30′	9.58284	9.96562	9.61722	0.38278	30′
45′	9.58739	9.96483	9.62256	0.37744	67° 15′
	log cos	log sin	log cot	log tan	

TABLE XL (continued)

LOGARITHMS OF TRIGONOMETRIC FUNCTIONS

(10 is to be subtracted from the number in the table.)

(For angles over 45°, use the numbers on the right and the functions below.)

READ DOWN	log sin	log cos	log tan	log cot	READ UP
23°	9.59188	9.96403	9.62785	0.37215	**67°**
15′	9.59632	9.96322	9.63310	0.36690	**45′**
30′	9.60070	9.96240	9.63830	0.36170	**30′**
45′	9.60503	9.96157	9.64346	0.35654	**15′**
24°	9.60931	9.96073	9.64858	0.35142	**66°**
15′	9.61354	9.95988	9.65366	0.34634	**45′**
30′	9.61773	9.95902	9.65870	0.34130	**30′**
45′	9.62186	9.95815	9.66371	0.33629	**15′**
25°	9.62595	9.95728	9.66867	0.33133	**65°**
15′	9.62999	9.95639	9.67360	0.32640	**45′**
30′	9.63398	9.95549	9.67850	0.32150	**30′**
45′	9.63794	9.95458	9.68336	0.31664	**15′**
26°	9.64184	9.95366	9.68818	0.31182	**64°**
15′	9.64571	9.95273	9.69298	0.30702	**45′**
30′	9.64953	9.95179	9.69774	0.30226	**30′**
45′	9.65331	9.95084	9.70247	0.29753	**15′**
27°	9.65705	9.94988	9.70717	0.29283	**63°**
15′	9.66075	9.94891	9.71184	0.28816	**45′**
30′	9.66441	9.94793	9.71648	0.28352	**30′**
45′	9.66803	9.94694	9.72109	0.27891	**15′**
28°	9.67161	9.94593	9.72567	0.27433	**62°**
15′	9.67515	9.94492	9.73023	0.26977	**45′**
30′	9.67866	9.94390	9.73476	0.26524	**30′**
45′	9.68213	9.94286	9.73927	0.26073	**15′**
29°	9.68557	9.94182	9.74375	0.25625	**61°**
15′	9.68897	9.94076	9.74821	0.25179	**45′**
30′	9.69234	9.93970	9.75264	0.24736	**30′**
45′	9.69567	9.93862	9.75705	0.24295	**15′**
30°	9.69897	9.93753	9.76144	0.23856	**60°**
15′	9.70224	9.93643	9.76580	0.23420	**45′**
30′	9.70547	9.93532	9.77015	0.22985	**30′**
45′	9.70867	9.93420	9.77447	0.22553	**15′**
31°	9.71184	9.93307	9.77877	0.22123	**59°**
15′	9.71498	9.93192	9.78306	0.21694	**45′**
30′	9.71809	9.93077	9.78732	0.21268	**30′**
45′	9.72116	9.92960	9.79156	0.20844	**15′**
32°	9.72421	9.92842	9.79579	0.20421	**58°**
15′	9.72723	9.92723	9.80000	0.20000	**45′**
30′	9.73022	9.92603	9.80419	0.19581	**30′**
45′	9.73318	9.92482	9.80836	0.19164	**15′**
33°	9.73611	9.92359	9.81252	0.18748	**57°**
15′	9.73901	9.92235	9.81666	0.18334	**45′**
30′	9.74189	9.92111	9.82078	0.17922	**30′**
45′	9.74474	9.91985	9.82489	0.17511	**15′**
34°	9 74756	9.91857	9.82899	0.17101	**56°**
15′	9 75036	9.91729	9.83307	0.16693	**45′**
30′	9 75313	9.91599	9.83713	0.16287	**30′**
45′	9.75587	9.91469	9.84119	0.15881	**55° 15′**
	log cos	log sin	log cot	log tan	

TABLE XL (continued)

LOGARITHMS OF TRIGONOMETRIC FUNCTIONS

(10 is to be subtracted from the number in the table.)

(For angles over 45°, use the numbers on the right and the functions below.)

READ DOWN	log sin	log cos	log tan	log cot	READ UP
35°	9.75859	9.91336	9.84523	0.15477	**55°**
15′	9.76129	9.91203	9.84925	0.15075	**45′**
30′	9.76395	9.91069	9.85327	0.14673	**30′**
45′	9.76660	9.90933	9.85727	0.14273	**15′**
36°	9.76922	9.90796	9.86126	0.13874	**54°**
15′	9.77181	9.90657	9.86524	0.13476	**45′**
30′	9.77439	9.90518	9.86921	0.13079	**30′**
45′	9.77694	9.90377	9.87317	0.12683	**15′**
37°	9.77946	9.90235	9.87711	0.12289	**53°**
15′	9.78197	9.90091	9.88105	0.11895	**45′**
30′	9.78445	9.89947	9.88498	0.11502	**30′**
45′	9.78691	9.89801	9.88890	0.11110	**15′**
38°	9.78934	9.89653	9.89281	0.10719	**52°**
15′	9.79176	9.89504	9.89671	0.10329	**45′**
30′	9.79415	9.89354	9.90061	0.09939	**30′**
45′	9.79652	9.89203	9.90449	0.09551	**15′**
39°	9.79887	9.89050	9.90837	0.09163	**51°**
15′	9.80120	9.88896	9.91224	0.08776	**45′**
30′	9.80351	9.88741	9.91610	0.08390	**30′**
45′	9.80580	9.88584	9.91996	0.08004	**15′**
40°	9.80807	9.88425	9.92381	0.07619	**50°**
15′	9.81032	9.88266	9.92766	0.07234	**45′**
30′	9.81254	9.88105	9.93150	0.06850	**30′**
45′	9.81475	9.87942	9.93533	0.06467	**15′**
41°	9.81694	9.87778	9.93916	0.06084	**49°**
15′	9.81911	9.87613	9.94299	0.05701	**45′**
30′	9.82126	9.87446	9.94681	0.05319	**30′**
45′	9.82340	9.87277	9.95062	0.04938	**15′**
42°	9.82551	9.87107	9.95444	0.04556	**48°**
15′	9.82761	9.86936	9.95825	0.04175	**45′**
30′	9.82968	9.86763	9.96205	0.03795	**30′**
45′	9.83174	9.86589	9.96586	0.03414	**15′**
43°	9.83378	9.86413	9.96966	0.03034	**47°**
15′	9.83581	9.86235	9.97345	0.02655	**45′**
30′	9.83781	9.86056	9.97725	0.02275	**30′**
45′	9.83980	9.85876	9.98104	0.01896	**15′**
44°	9.84177	9.85693	9.98484	0.01516	**46°**
15′	9.84373	9.85510	9.98863	0.01137	**45′**
30′	9.84566	9.85324	9.99242	0.00758	**30′**
45′	9.84758	9.85137	9.99621	0.00379	**15′**
45°	9.84949	9.84949	0.00000	0.00000	**45°**
	log cos	**log sin**	**log cot**	**log tan**	

Glossary of Mathematical Terms

acceleration: the rate of change of speed with respect to time.

adjacent side: in a right triangle, the side which, with the *hypotenuse*, forms the angle under discussion. (See page 406.)

angular velocity: the rate at which the angle between a fixed line and the line joining the moving object to a fixed point changes. (See page 390.)

co-function: the corresponding *function* of the *complement*.

cosecant: the *function* of the angle which represents the value of the *hypotenuse* divided by the *opposite side*. (See pages 394, 399ff.) The *reciprocal* of the *sine*.

cosine: the *function* of the angle which represents the value of the *adjacent side* divided by the *hypotenuse*. (See pages 394, 399ff, 414.) The *reciprocal* is the *secant*.

cotangent: the *function* of the angle which represents the value of the *adjacent side* divided by the *opposite side*. (See pages 394, 399ff.) The *reciprocal* of the *tangent*.

depression, angle of: the angle between a *horizontal* line (imagined) and the *oblique* line which joins the observer's eye to an object lower than the line of the eye. (See page 408.)

double-angle: an angle of twice the value of a given angle. (See page 424.)

elevation, angle of: the angle between a *horizontal* line (imagined) and the *oblique* line which joins the observer's eye to an object above the line of the eye. (See page 408.)

function: in trigonometry, the ratio of the sides of the angle of a right triangle. (See pages 394, 399ff.)

half-angle: an angle of half the value of a given angle. (See page 424.)

identities: a statement of the relationship of trigonometric *functions*. (See page 418ff.)

inertia: rest; resistance to change; the property of a body which requires exertion of force to give *acceleration* to the body. (See page 390.)

initial (position): the original position. (See page 386.)

interpolation: computation of a value not given in a table. (See page 95.)

inverse function: the *function* whose value is the angle whose trigonometric *function* is given in the statement. (See page 427.)

kinetic energy: the energy or capacity for doing work which a body possesses by virtue of its motion. (See page 350.)

mil: a unit for measuring angles, equivalent to $\frac{1}{6400}$ of a complete *revolution*. (See pages 391, 440.)

milliradian: one one-thousandth of a *radian*. (See page 389.)

minute: $\frac{1}{60}$ of a *degree*. (See pages 318, 387, 440.)

multiple angle: an angle which is a definite number of times the size of a given angle. (See page 424.)

oblique triangle: an angle containing an *oblique* angle.

opposite side: in a right triangle, the side opposite the angle under discussion. (See page 406.)

pi (π): the symbol denoting the relationship between the *circumference* and the *diameter* (or *radius*) of a circle. (See page 388.)

quadrant: the section of a graph between the *axes* of *coördinates*. The upper-right-hand quadrant is designated as I; the others are numbered consecutively in counterclockwise order. (See page 392.)

radian: a central angle subtended by an *arc* which is the same length as the *radius* of the circle. (See page 387.)

reciprocal: (See page 192.)

reduction: the expression of a *function* of an angle greater than 90° in terms of an angle of 90° or less. (See page 385.)

revolution: the measurement of an angle whose *initial* and *terminal* sides *coincide*, 360°. (See page 395.)

rotation: the movement of the *initial* side of the angle to its *terminal* position. (See page 385.)

secant: the *function* of the angle which represents the value of the *hypotenuse* divided by the *adjacent side*. (See pages 394, 399ff.) The *reciprocal* of the *cosine*.

second: $\frac{1}{60}$ of a degree. (See pages 387, 440.)

sine: the *function* of the angle which represents the value of the *opposite side* divided by the *hypotenuse*. (See pages 394, 399ff, 411.) The *reciprocal* is the *cosecant*.

tangent: the *function* of the angle which represents the value of the *opposite side* divided by the *adjacent side*. (See pages 394, 399ff, 417.) The *reciprocal* is the *cotangent*.

terminal (side): the position of the side of the angle after it has been *rotated* from its *initial* position. (See page 386.)

trajectory: the path of a moving object. (See page 385.)

trigon: another name for *triangle*. (See page 385.)

trigonometry: the study of the *ratios* (and relationship of ratios) between the sides of a triangle and their use in finding the remaining parts when a sufficient number of parts are known.

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